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TABLE OF CONTENTS

ABSTRACT

We develop a geometric approach to algebras in congruence modular varieties. The idea of coordinatization of lines in affine geometry finds an almost perfect analog in the coordinatization of algebras.
The geometry is the congruence class geometry, i.e. the subspaces are the blocks of congruence relations.

We show that congruence modularity guarantees that the congruence class geometry behaves nicely, because the Desarguesian and the Pappian theorems are true, if interpreted correctly. The innocuously looking "Shifting Lemma" is the basic and powerful tool we need.
The obstacle to a perfect coordinatization is a congruence relation called the "commutator". The commutator is zero iff nonparallel lines have precisely one point of intersection.
This approach leads to a simple geometric development of commutator theory for arbitrary congruences. Results about affine algebras on the one hand and about distributive varieties on the other hand are tied together where only the commutator appears as a parameter. For the extreme values of this parameter we find theorems about affine, nilpotent and solvable congruences and varieties at one end and theorems generalizing Jônsson's lemma at the other end. A radical, $\sqrt{\underline{A}}$, is defined and we show that Jônsson's lemma is true for every algebra $\underline{A}/\sqrt{\underline{A}}$.

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PREFACE

Affine planes can be coordinatized by certain algebraic structures called planar ternary rings. If the Desarguesian theorem holds, then a (generally noncommutative) field is obtained. The algebraic structure can then be used in turn to prove results which may be translated back into geometrical theorems. The proof that every finite Desarguesian affine plane is Pappian, using Wedderburn's theorem is a prominent example.

By the same token geometrical reasoning may assist the mathematician working on algebraic structures. In theories related to linear algebra, as for example in ring - and module theory geometric intuition may suggest algebraic results and may be a guide to algebraic proofs. This method is not limited to fields that essentially originated from geometry. Here we deal with classes of algebraic structures, general enough to include groups, rings and modules on the one hand and lattices on the other hand, but they do not include semigroups, for example. We shall see that thereby we seem to have found the right level of abstraction where a geometrical language may reasonably be used and a geometrical intuition may be developped.

The key is that every (universal) algebra coordinatizes a pseudo-geometry. This geometry was investigated and characterized by Wille in [41], he called it the "Kongruenzklassengeometrie". It may be a geometry of a rather strange nature, but the fundamental notions like "points", "lines" and "incidence" make sense, so as to allow us to draw pictures of geometrical configurations, corresponding to algebraic contexts. These configurations may then lead to a deeper understanding of the theory, but also suggest proofs, which then have to be reformulated algebraically.

Here we make extensive use of this geometric visualization. We draw points and lines to express and explain algebraic situations. Previously known theories become clearer and new results are obtained. Indeed, by a consequent use of the geometric method, some deep algebraic results may seem more obvious because they are suggested by the geometry.

As we have mentioned before, the geometry may be very nasty in general, and it was long believed that only the class of "permutable" varieties was satisfactorily tractable by geometric methods. Those algebras share the fundamental property that "parallelograms" in their Kongruenzklassengeometrie can be completed, i.e. given three points, there exists a least one fourth point, so that the four points form a parallelogram. Those algebras comprise all classical structures mentioned above, so certainly they might seem to form a reasonable level of abstraction. On the other hand, one would like to include some more classes of structures into a systematic treatment. For example, the class of all lattices which does not have this property.

The framework we have chosen is modularity. We assume that the algebras in question have modular congruence lattices. This class is known since Birkhoff [2] to include permutable varieties, and clearly lattices are captured too, since in fact their congruence lattices are distributive.

Thus the difficulty is to find the right geometric properties common to both kinds of theories. A very simple property, the "Shifting Lemma" was discovered in [17] and its importance has become more apparent henceforth. In fact, more-dimensional analogues like the "Little Desarguesian theorem" and the "Escher-Cube" could be proved [18], showing the richness of structure in modular congruence class spaces.

There is a broad spectrum of modular varieties, reaching from abelian groups and modules at the one end to lattices or more generally to distributive varieties at the other end. Moreover, given a modular variety then after imposing appropriate conditions one often finds that either the variety is affine (i.e. polynomially equivalent to a variety of modules) or it has typical features of a distributive variety. This dichotomy was first noticed for permutable varieties.

Now, as we have said before, lattices and abelian groups seem to be modular out of different reasons and in fact they turn out as different ends in a spectrum of modular varieties which lie in between. In fact those ends could be isolated inside the subclass of permutable varieties. As an example we mention R. McKenzie's results [33] and also [16]. It was only when J.D.H. Smith [36] introduced "commutators" into general algebra, that an overview over the subspectrum of permutable varieties could be handled. Commutators in a way acted like a prism, making the spectrum visible.

J. Hagemann and C. Herrmann [24] managed to carry the commutator concept over to modular varieties, proving many of its properties that now seem to be fundamental for the theory. The sacrifice, however, was the loss of geometric intuition, the complicacy of the concept, which made it extremely hard to handle.

Using the Shifting Lemma, a very simplified definition could be given in [19], making it possible to give an elementary and geometrical development of commutator theory. The theory has since been proven extremely useful, pushing the theory of modular varieties forcefully ahead. Most prominently R. McKenzie's work with R. Freese [13] and with S. Burris [6] has to be mentioned here. Progress in other directions was also made in [21].

Let us now give a brief overview to the present treatise. After establishing the fundamental concepts and notions in chapter 0, we introduce the reader to the concept of modularity in chapter 1. The congruence class geometry is developed in chapter 2, we prove the fundamental configuration theorems, like the Little Desarguesian theorem, the Cube Lemma and the

Escher-Cube Lemma which will reappear throughout these notes. The syntactical description of modularity due to Day [9] follows in chapter 3, we try to explain the geometry behind it. Chapter 4 provides a term which plays a fundamental rôle in modular varieties. The door is opened to carry results from permutable varieties over to modular varieties. The construction of the aforementioned term is an outstanding example of how geometry provides the ideas for an algebraic proof. In chapter 5 we show that the classical idea of coordinatizing the affine plane with an abelian group has a direct counterpart in modular varieties. An analysis of the ingredients leads to the investigation of commutators in chapter 6, the theory is developed and the properties that nowadays seem to be fundamental are proved. The famous Jónsson Lemma turns out to be a special instance of a more general theorem for modular varieties. A radical $\sqrt{\underline{A}}$ is defined and it is shown that Jónssons Lemma is true for every algebra of the form $\underline{A}/\sqrt{\underline{A}}$ in a modular varietiey. A formula which shows that congruences permute modulo some commutators allows us to give a Mal'cev type description of modular varieties using ternary terms only in chapter 7. More evidence is gathered that modularity lives between the poles of permutability and distributivity. Theorems that refer to permutability are proved in chapter 8. Abelian congruences and corresponding affine algebras are then examined in chapter 9 and nilpotent and solvable varieties characterized in chapter 10. In chapter 11 we look at the possibility of yet extending the framework of modular varieties. FP-varieties seem to be appropriate for many results. Their congruence class geometry has special properties only in direct products of algebras. Terms being "n-ary homomorphisms" with respect to other terms are studied in chapter 12, a theme which is intimately connected with commutators and coordinatization. Unitary groupoid objects in modular varieties (and in FP-varieties) are shown to be abelian group objects.

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0. FUNDAMENTAL CONCEPTS

This chapter summarizes fundamental notions and elementary results of general algebra. We omit proofs since they can be found in most elementary textbooks, such as GRÄTZER [14], or PIERCE [43].

Regarding our terminology we mainly follow [14]. Our word "polynomial", however, stands for "algebraic function" as in [14]. We will in fact use both words simultaneously.

Let $\Delta := (n_i)_{i\in I}$ be a family of natural numbers. An <u>algebra of type Δ</u> is a pair $\underline{A} := (A,(f_i)_{i\in I})$ where A is a nonempty set and every f_i is an n_i-ary operation on A, i.e. a map $f_i\colon A^{n_i} \to A$.

From now on we tacitly assume we have specified a type Δ so that all algebras we deal with are of type Δ.

Let $\underline{B}$ be another algebra, i.e. $\underline{B} = (B,(g_i)_{i\in I})$.
A map $\phi\colon A \to B$ is a <u>homomorphism</u> from $\underline{A}$ to $\underline{B}$ if for every $i\in I$ and elements $a_1,a_2,\dots,a_{n_i} \in A$ we have

$$\phi(f_i(a_1,\dots,a_{n_i})) = g_i(\phi(a_1),\dots,\phi(a_{n_i})).$$

If ϕ is injective it is called an embedding. If ϕ is bijective then ϕ^{-1} is a homomorphism from $\underline{B}$ to $\underline{A}$. ϕ is then called an isomorphism. We say $\underline{A}$ and $\underline{B}$ are isomorphic and write $\underline{A} \cong \underline{B}$.

A subalgebra $\underline{C}$ of $\underline{A}$ is an algebra $\underline{C} = (C,(h_i)_{i\in I})$ where $C \subseteq A$ and each operation h_i is the restriction of the corresponding f_i to C^{n_i}.

If D is a nonempty subset of A and for all $d_1,\dots,d_{n_i} \in D$ we have that $f_i(d_1,\dots,d_{n_i}) \in D$ then $\underline{D} = (D,(h_i)_{i\in I})$ with $h_i := f_i|_{D^{n_i}}$ is a subalgebra of $\underline{A}$. By abuse of language we sometimes phrase shortly: D is a subalgebra of $\underline{A}$.

The <u>product</u> $\prod_{j\in J} \underline{C}_j$ of the algebras $\underline{C}_j$ has as underlying set the cartesian product of the sets C_j and the operations are defined componentwise. A subalgebra $\underline{S}$ of $\prod_{j\in J} \underline{C}_j$ is called a <u>subdirect product</u> of the $\underline{C}_j$, if the restrictions of the canonical projections π_j to $\underline{S}$ are still onto.

A <u>congruence relation</u> Θ on the algebra $\underline{A}$ is an equivalence relation on A which is at the same time a subalgebra of $\underline{A} \times \underline{A}$. For $(x,y) \in \Theta$ we frequently write $x\,\Theta\,y$ or $x \equiv y \pmod{\Theta}$. The fact that Θ is a subal-

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gebra of $\underline{A} \times \underline{A}$ can be expressed by the implication

$$x_1 \theta y_1, \ldots, x_{n_i} \theta y_{n_i} \Rightarrow f_i(x_1,\ldots,x_{n_i}) \,\theta\, f_i(y_1,\ldots,y_{n_i}).$$

This property is often referred to as the <u>compatibility</u> of θ with the operation f_i.

If θ is a congruence relation on $\underline{A}$ and $a \in \underline{A}$ we define $[a]\theta := \{x \in A \mid x \,\theta\, a\}$ and call it the <u>θ-block</u> of a.

The set $A/\theta := \{[a]\theta \mid a \in A\}$ can be given the structure of an algebra again by defining

$$f_i([a_1]\theta, \ldots, [a_{n_i}]\theta) := [f(a_1,\ldots,a_{n_i})]\theta.$$

The resulting algebra is called the <u>factor</u> of $\underline{A}$ by θ and

$\pi_\theta: A \to A/\theta$
$a \to [a]\theta$ is a surjective homomorphism.

Note that for an arbitrary homomorphism $\phi: \underline{A} \to \underline{B}$ the relation $\ker \phi := \{(x,y) \mid \phi(x) = \phi(y)\}$ is a congruence relation and every congruence relation arises this way, namely $\theta = \ker \pi_\theta$. Moreover, if $\phi: \underline{A} \to \underline{B}$ is a surjective homomorphism then $\underline{B}$ and $\underline{A}/\ker \phi$ are isomorphic, in symbols $\underline{B} \cong \underline{A}/\ker \phi$.

Let now $\underline{V}$ be a class of algebras. $\underline{V}$ is called a <u>variety</u> if $\underline{V}$ is closed under the formation of subalgebras, homomorphic images and direct products of any of its members. G. BIRKHOFF's theorem says that a class of algebras is definable by equations if and only if it is a variety.

In a variety $\underline{V}$ there exist free algebras $\underline{F}_{\underline{V}}(X)$ for every set X. $\underline{F}_{\underline{V}}$ is a functor, left adjoint to the forgetful functor into sets, in particular: For every algebra $\underline{A}$ in $\underline{V}$ and every map $\alpha: X \to A$ there exists precisely one homomorphism $\overline{\alpha}: \underline{F}_{\underline{V}}(X) \to \underline{A}$ extending α.

If X is finite, say $X = \{x_1,\ldots,x_k\}$ then $\underline{F}_{\underline{V}}(X)$ can be considered as the algebra of all k-ary <u>V-terms</u> with variables from X in the following manner:

For $p \in \underline{F}_{\underline{V}}(X)$ and $\underline{A} \in \underline{V}$ define a k-ary operation $p^{\underline{A}}$ on A by $p^{\underline{A}}(a_1,\ldots,a_k) := \overline{\alpha}(p)$, where $\overline{\alpha}$ is the unique homomorphism from $\underline{F}_{\underline{V}}(X)$ to $\underline{A}$ with $\overline{\alpha}(x_i) = a_i$.

The k-ary operations thus arising on A are called <u>term-functions</u>. They can also be characterized as those k-ary operations on A which can be built up by superposition from the fundamental operations f_i and the projection operations $\pi_i^n: A^n \to A$, $(a_1,\ldots,a_n) \to a_i$.

If $n-k$ places in an n-ary term function are frozen with fixed elements of A, we obtain a k-ary <u>polynomial</u> of $\underline{A}$, frequently also called an <u>algebraic function</u> of $\underline{A}$.

We will have to take a closer look at congruence relations. If θ is a congruence relation on $\underline{A}$ and τ a polynomial of $\underline{A}$ then Θ is compatible with τ. (Similarly the equality characterizing homomorphisms remains true if f_i is replaced by any term-function.)

There are two trivial congruences 0 and 1 (sometimes subscripted as $0_{\underline{A}}$ and $1_{\underline{A}}$) on every algebra $\underline{A}$, given by

$$0 = \{(x,x) \mid x \in \underline{A}\} \text{ and}$$
$$1 = \{(x,y) \mid x,y \in \underline{A}\}.$$

$\underline{A}$ is called <u>simple</u> if there are no other congruences on $\underline{A}$.
$\underline{A}$ is called <u>subdirectly irreducible</u>, if $\underline{A}$ possesses a smallest nontrivial congruence relation μ, called the <u>monolith</u> of $\underline{A}$. G. BIRKHOFF's theorem asserts that every algebra is a subdirect product of subdirectly irreducible factors.

The intersection of an arbitrary family of congruences is a congruence again, thus for a subset $T \subseteq A \times A$ there exists a smallest congruence relation containing T which we will denote by $\langle T \rangle_{\underline{A}}$ or $\Theta_{(a,b)}$ for $T = \{(a,b)\}$.

A.I. MAL'CEV in [32] gave an explicit description of $\langle T \rangle_{\underline{A}}$. This description specializes as follows:

0.1 <u>Theorem:</u> (i) If T is a reflexive symmetric relation, then $(a,b) \in \langle T \rangle_{\underline{A}}$ iff there exist unary algebraic functions $\tau_0,\dots,\tau_n$ and $(s_0,t_0),\dots,(s_n,t_n) \in T$ such that

$$a = \tau_0(s_0)$$
$$\tau_i(t_i) = \tau_{i+1}(s_{i+1}) \quad \text{for } 0 \le i < n,$$
$$\tau_n(t_n) = b$$

(ii) For $T = \{(x,y)\}$: $(a,b) \in \Theta_{(x,y)}$ iff there exist unary algebraic functions $\tau_0,\dots,\tau_{n-1}$ and elements $a = c_0,\dots,c_{n-1}$, $c_n = b$ such that $\{c_i,c_{i+1}\} = \{\tau_i(x),\tau_i(y)\}$ for $0 \le i < n$.

(iii) A subset $S \subseteq A$ is a class of some congruence relation on $\underline{A}$ iff for all algebraic functions τ on $\underline{A}$ we have either $\tau(S) \subseteq S$ or $\tau(S) \cap S = \emptyset$.

The congruences on an algebra form a complete algebraic lattice, in fact a sublattice of the lattice of all equivalence relations on the set A. We shall denote this lattice by $\mathrm{Con}(\underline{A})$. The join of two congruences Θ and Ψ always contains the relational product $\Theta \circ \Psi$ and can be described as

$$\Theta \vee \Psi = \bigcup \{\overbrace{\Theta \circ \Psi \circ \Theta \circ \Psi \circ \dots \circ \Theta \circ \Psi}^{n \text{ times}} \mid n \in \mathbb{N}\}.$$

Since $\Theta \circ \Theta = \Theta$ by transitivity, it follows that $\Theta \vee \Psi = \Theta \circ \Psi$ just in case that $\Theta \circ \Psi = \Psi \circ \Theta$. We shall then say that Θ and ψ are permutable.

Let now $\phi: \underline{A} \to \underline{B}$ be a homomorphism. If β is a congruence relation on $\underline{B}$ then

$$\overleftarrow{\phi}\beta := \{(x,y) \in A \times A \mid (\phi(x),\phi(y)) \in \beta\}$$

is a congruent relation on $\underline{A}$. Incidentally $\ker \phi = \overleftarrow{\phi}0$ and $\overleftarrow{\phi}$ is a lattice isomorphism from $\mathrm{Con}(\underline{B})$ to the sublattice of $\mathrm{Con}(\underline{A})$ consisting of all congruence relations on $\underline{A}$ which contain $\ker \phi$.

For $\alpha \in \mathrm{Con}(\underline{A})$ we get a congruence

$$\overrightarrow{\phi}\alpha := \langle\{(\phi(x),\phi(y)) \mid x \,\alpha\, y\}\rangle_{\underline{B}} \text{ on } \underline{B}.$$

One checks the relations

$$\overrightarrow{\phi}\overleftarrow{\phi}\beta \le \beta \quad \text{and}$$

$$\overleftarrow{\phi}\overrightarrow{\phi}\alpha \ge \alpha \vee \ker \phi$$

with equality holding in both formulas if ϕ is onto.

If $\prod_{j \in J} \underline{C}_j$ is a direct product, then the kernels of the canonical projections will also be denoted as π_j. Those congruences are often called factor congruences. They are mutually permutable. Given a filter $\mathcal{D}$ on the indexing set J, a congruence relation $\Theta_{\mathcal{D}}$ arises on defining: $a \,\Theta_{\mathcal{D}}\, b$ iff $\{i \in J \mid \pi_i(a) = \pi_i(b)\} \in \mathcal{D}$. If $\mathcal{D}$ is an ultrafilter, the important construction of an ultraproduct $\prod_{i \in J} \underline{C}_j / \Theta_{\mathcal{D}}$ is obtained. It has important modeltheoretic properties, see e.g. [1].

1. MODULARITY

1.1 Definition: A lattice $\underline{L}$ is modular if for every $x,y,z \in L$ the implication

$$x \geq z \Rightarrow x \wedge (y \vee z) \leq (x \wedge y) \vee z$$

holds.

An algebra $\underline{A}$ is called congruence modular if $\mathrm{Con}(\underline{A})$ is a modular lattice. Similarly a variety $\underline{V}$ is called modular if every $\underline{A} \in \underline{V}$ is congruence modular.

The following theorems will be used to provide us with a sufficiently large class of examples.

The first theorem is again due to MAL'CEV. It describes congruence permutable varieties, i.e. varieties all of whose algebras are congruence permutable.

1.2 Theorem [32]: A variety $\underline{V}$ of algebras is congruence permutable if and only if there exists a ternary $\underline{V}$-term $p(x,y,z)$, such that the equations $p(x,y,y) = x$ and $p(x,x,y) = y$ are true in $\underline{V}$.

To generalize the concept of permutability of congruences we define for a natural number k and congruences Θ_o and Θ_1:

Definition: Congruences Θ_o and Θ_1 are k-permutable if $\Theta_o \circ \Theta_1 \circ \Theta_o \circ \ldots \circ \Theta_\varepsilon = \Theta_1 \circ \Theta_o \circ \Theta_1 \circ \ldots \circ \Theta_{1-\varepsilon}$, where ε is 1 or o, depending on whether k is even or odd, and both sides are k-fold relational products. An algebra is called k-permutable, if any two congruences on $\underline{A}$ are k-permutable. Similarly a variety consisting of k-permutable algebras only will be called a k-permutable variety.

Thus permutability is just 2-permutability and k-permutability implies (k+1)-permutability.

J. HAGEMANN and A. MITSCHKE obtained a characterization of k-permutable varieties reminiscent of MAL'CEV's theorem, and including this for the case $k = 2$ [25]:

1.3 Theorem: A variety $\underline{V}$ of algebras is k-permutable if and only if there exist ternary $\underline{V}$-terms $p_o,\ldots,p_k$ such that the equations

$$p_o(x,y,z) = x$$
$$p_i(x,x,y) = p_{i+1}(x,y,y) \quad \text{for } 0 \leq i < k$$
$$p_k(x,y,z) = z$$

are true in $\underline{V}$.

To see that 3-permutable algebras are congruence modular, first shown by B. JONSSON [29], we note that joins of two congruences Θ and Ψ in this case are computed as $\Theta\circ\Psi\circ\Theta$. Here the modular law reduces to $\alpha \geq \gamma \Rightarrow \alpha \wedge (\gamma\circ\beta\circ\gamma) \leq (\alpha \wedge \beta) \vee \gamma$. For (x,y) from the left hand side there exist u,v with $x\ \gamma\ u\ \beta\ v\ \gamma\ y$ and consequently $x\ \alpha\ u$ and $v\ \alpha\ y$. α is transitive, yielding $u\ \alpha\ v$. Finally $x\ \gamma\ u\ (\alpha \wedge \beta)\ v\ \gamma\ y$ hence $(x,y) \in (\alpha \wedge \beta) \vee \gamma$.

Many examples of congruence modular algebras are in fact <u>congruence distributive</u>. B. JONSSON characterized varieties containing congruence distributive algebras only, as follows:

1.4 <u>Theorem ([28]):</u> A variety $\underline{V}$ of algebras is congruence distributive if and only if for some natural number n there exist ternary $\underline{V}$-terms $q_o,\dots,q_n$ such that the equations

$$q_o(x,y,z) = x$$
$$q_i(x,y,x) = x \qquad \text{for all } 0 \leq i \leq n,$$
$$q_i(x,x,y) = q_{i+1}(x,x,y) \qquad \text{for } 0 \leq i < n, \ i \text{ even.}$$
$$q_i(x,y,y) = q_{i+1}(x,y,y) \qquad \text{for } 0 < i < n, \ i \text{ odd}$$
$$q_n(x,y,z) = z$$

hold in $\underline{V}$.

Now we are able to list many varieties which are congruence modular:

<u>Groups:</u> Groups are permutable according to MAL'CEV's theorem. $P(x,y,z) := x\cdot y^{-1}\cdot z$ is the term witnessing permutability.

<u>Rings:</u> Same as above with $p(x,y,z) = x-y+z$.

<u>Quasigroups:</u> $p(x,y,z) := (x/(y\backslash y))(y\backslash z)$ is a term for MAL'CEV's theorem.

<u>Median algebras:</u> Those are algebras with a ternary "majority term" i.e. a term $m(x,y,z)$ satisfying the equations $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$. Median algebras are congruence distributive, JONSSON's theorem applies with $n = 2$.

<u>Lattices:</u> Lattices are median algebras. Take $m(x,y,z) := (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$.

<u>Implication algebras:</u> They are groupoids $(G,\rightarrow)$ satisfying $(x\rightarrow y)\rightarrow x = x$; $(x\rightarrow y)\rightarrow y = (y\rightarrow x)\rightarrow x$; $x \rightarrow (y\rightarrow z) = y \rightarrow (x\rightarrow z)$. Implication algebras are 3-permutable and congruence distributive. See MITSCHKE [34] and HAGEMANN, MITSCHKE [25].

<u>Boolean algebras:</u> Boolean algebras are lattices, hence congruence distributive. But they are also rings, hence permutable.

So far all of our examples were already either permutable or distributive. For an example of a congruence modular variety which is neither permutable, nor congruence distributive we introduce:

<u>Generalized right complemented semigroups:</u> These algebras have two binary operations $\cdot$ and $*$, satisfying:

$$x\cdot(x*y) = y\cdot(y*x)$$
$$x\cdot(y*y) = x.$$

Generalized right complemented semigroups have 3-permutable congruences. HAGEMANN and MITSCHKE showed that their theorem applies with $p_1(x,y,z) = x\cdot(y*z)$ and $p_2(x,y,z) = z\cdot(y*x)$.

To see that generalized right complemented semigroups are in general non-distributive, take a ring $\underline{R}$ with unit, in which 2 has an inverse. Then define $x\cdot y := x+y$ and $x*y := \frac{1}{2}(y-x)$ as operations on any module over $\underline{R}$.

Implication algebras are also models of the above equations. If we define $x\cdot y := y\to x$ and $x*y := y\to x$ then the above equations hold. See MITSCHKE [34] for an example of a (three-element) implication algebra with non permutable congruences.

If a further equation

$$(x\cdot y)*z = y*(x*z)$$

is added, we obtain the class of right complemented semigroups. See BOSBACH [3] for the fact that right complemented semigroups are congruence distributive.

Also see the remarks at the end of chapter 7.

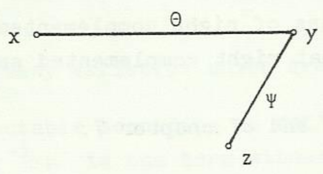
2. CONGRUENCE CLASS GEOMETRY

If V is a vector space then the blocks of congruence relations on V are precisely the affine subspaces of V. With this example in mind, geometrical terms suggest themselves for the study of congruence relations. Indeed the system of congruence classes of an algebra $\underline{A}$ can be considered as a geometry, the so called "Kongruenzklassengeometrie". This geometry was introduced and investigated by R. WILLE in [41].

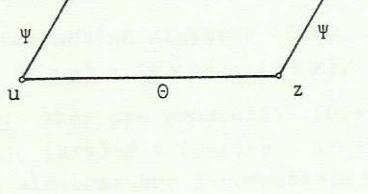
We do not use any results from this approach, yet we adopt and heavily exploit the geometrical viewpoint, by using a pseudogeometrical language and by drawing geometrical figures.

Thus we draw points for elements of a given algebra $\underline{A}$ and we connect two points, say x and y with a line, if x and y are congruent modulo some congruence relation (or some compatible relation), say Θ. In this case we label the line connecting x and y with the symbol Θ and think of it as representing all points from $[x]\Theta$ when Θ is a congruence relation. Two lines will be drawn parallel, just in case they are classes of the same congruence relation.

As an example, the following picture expresses the relation $x\ \Theta\ y\ \Psi\ z$:



Moreover we will have $\Theta\circ\Psi = \Psi\circ\Theta$ if and only if for every $x,y,z \in \underline{A}$, the above picture can be completed to



for some $u \in \underline{A}$.

Thus permutability of congruences can be expressed geometrically by the existence of the "4-th parallelogram point". (x,y,u,z) in this case would be called a <u>Θ-Ψ-parallelogram</u>. MAL'CEV's term $p(x,y,z)$ always provides us with one 4-th parallelogram point. From the equations $p(x,y,y) = x$ and $p(y,y,z) = z$ and the relations $x \; \Theta \; y \; \Psi \; z$ we infer $p(x,y,z) \; \Theta \; p(y,y,z) = z$ and $p(x,y,z) \; \Psi \; p(x,y,y) = x$.

In congruence modular algebras such a strong geometrical tool cannot be expected, however the following "Shifting Lemma" is still powerful enough to replace the modular equation in everything that follows.

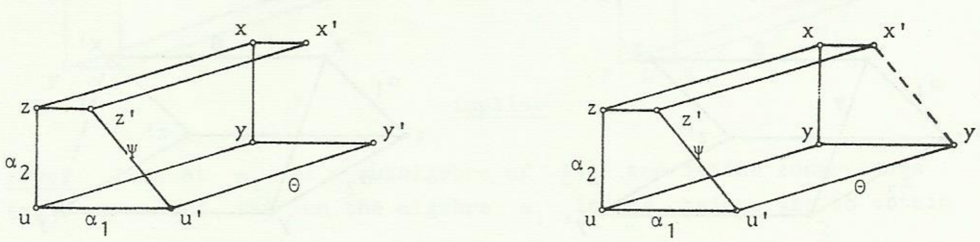
2.1 <u>Shifting Lemma:</u> Let $\alpha,\beta,$ and γ be congruences on a congruence modular algebra $\underline{A}$ and let x,y,z,u be elements of $\underline{A}$. If $\alpha \wedge \beta \leq \gamma$ then

<u>implies</u>

<u>Proof:</u> We have $(x,y) \in \alpha \wedge (\beta \circ (\alpha \wedge \gamma) \circ \beta) \subseteq \alpha \wedge (\beta \vee (\alpha \wedge \gamma)) \subseteq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ by modularity. From the assumption that $\alpha \wedge \beta \leq \gamma$ it follows that $(x,y) \in \gamma$. □

If the condition $\alpha \wedge \beta \leq \gamma$ is dropped in the Shifting Lemma the hypothesis still guarantees that $(x,y) \in (\alpha \wedge \beta) \vee \gamma$; simply replace γ by $\gamma' := (\alpha \wedge \beta) \vee \gamma$ and apply 2.1. Thus both versions are equivalent. We will find that within a variety the validity of the Shifting Lemma is equivalent to modularity. In fact, the Shifting Lemma will replace the modular law in everything that follows. In particular for the rest of this chapter we will assume that the Shifting Lemma holds in $\underline{A} \times \underline{A}$. Thus we are able to obtain "higher-dimensional" configuration theorems which are important for later chapters. Those theorems were found in GUMM [18]. Our original proof used the DAY-terms which we shall introduce in the following chapter. The proof was shortened by TAYLOR and by WOLF, to the form we present it in here.

2.2 <u>Theorem:</u> Let Θ,α_1,α_2 and Ψ be congruences with $\Theta \wedge \alpha_1 \leq \Psi \geq \Theta \wedge \alpha_2$. If x,y,z,u,x',y',z',u' are elements of $\underline{A}$ then



<u>implies</u>

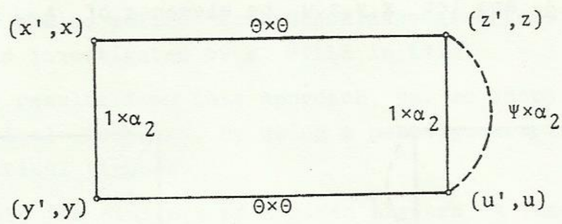
Proof: The pairs (x',x), (y',y), (z',z), (u',u) are elements of α_1 which is a subalgebra of $\underline{A}\times\underline{A}$. Defining congruences $1\times\alpha_2$ and $\Psi\times\alpha_2$ and $\Theta\times\Theta$ on α_1 by

$$(u,v)\ 1\times\alpha_2\ (r,s) \quad \text{iff} \quad v\ \alpha_2\ s$$

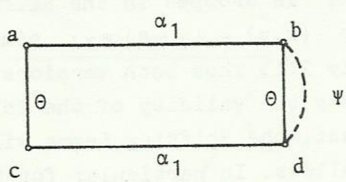
$$(u,v)\ \Psi\times\alpha_2\ (r,s) \quad \text{iff} \quad u\ \Psi\ r \ \text{ and } \ v\ \alpha_2\ s$$

$$(u,v)\ \Theta\times\Theta\ (r,s) \quad \text{iff} \quad u\ \Theta\ r \ \text{ and } \ v\ \Theta\ s$$

we obtain



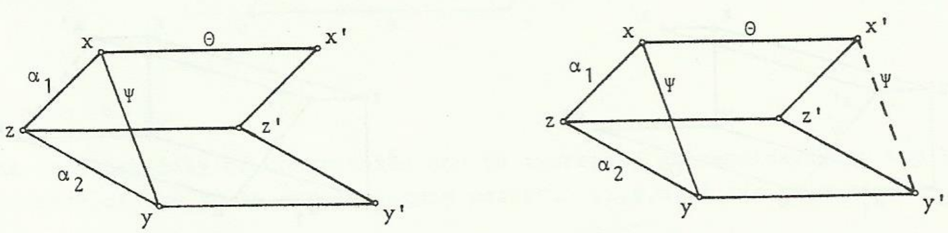
If we are allowed to apply the Shifting Lemma, then from $(x',x)\ \Psi\times\alpha_2\ (y',y)$ we get immediately $x'\ \Psi\ y'$ and we are done. However, we have to check that $\Theta\times\Theta \wedge 1\times\alpha_2 \leq \Psi\times\alpha_2$. Thus let $(a,b)\ \Theta\times\Theta \wedge 1\times\alpha_2\ (c,d)$. First of all $(a,b)\in\alpha_1$ and $(c,d)\in\alpha_1$. Moreover $(b,d)\in\Theta\wedge\alpha_2$ hence $(b,d)\in\Psi$ by assumption. Hence



The other assumption, $\Theta\wedge\alpha_1 \leq \Psi$ allows us to apply the Shifting Lemma in this situation, yielding $(a,c)\in\Psi$.

Mainly we are interested in the following two special cases. Firstly, letting $x = x'$ and $z = z'$ we obtain

2.3 The Little Desarguesian Theorem: Let Θ, α_1, α_2 and Ψ be congruences with $\Theta\wedge\alpha_1 \leq \Psi \geq \Theta\wedge\alpha_2$. Then



implies

This theorem is particularly interesting, since R. FREESE and B. JONSSON [11] have shown that the congruence lattices of algebras in modular varieties are arguesian. Thus our theorem may be considered as an affine counterpart to their result. The existence of such an affine counterpart is surprising since no transition between the projective geometry of algebras (as manifested in their congruence lattices) and the affine geometry (Kongruenzklassengeometrie) is known.

Our next specialization of 2.2 has many applications in what follows. In particular it guarantees the closure of the "REIDEMEISTER configuration" which was first shown and applied in [16] and [17].

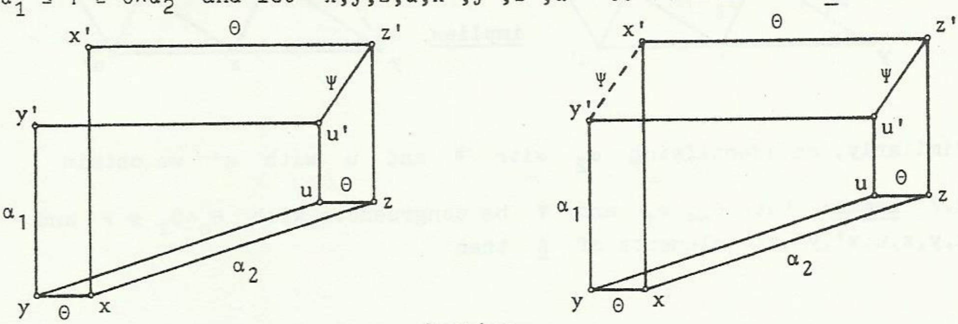
2.4 <u>Cube Lemma:</u> Let Θ_o, Θ_1 and Ψ be congruences with $\Theta_o \wedge \Theta_1 \leq \Psi$ and let x,y,z,u,x',y',z',u' be elements of $\underline{A}$, then

<u>implies</u>

<u>Proof:</u> Set $\Psi = \alpha_2$, $\Theta_o := \Theta$, $\Theta_1 := \alpha_1$ in 2.2. □

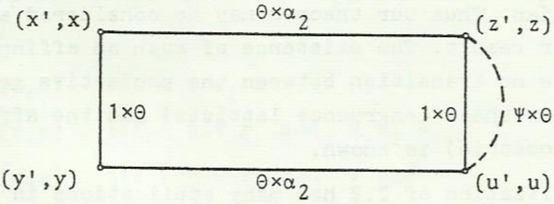
A "twisted" form of 2.2 will be needed to give us the closure of the "Desarguesian configuration", as termed in [17]:
Because of apparent common features of the configuration with pictures of M.C. ESCHER we call it

2.5 <u>The ESCHER Cube:</u> Let Θ, α_1, α_2 and Ψ be congruences with $\Theta\wedge\alpha_1 \leq \Psi \geq \Theta\wedge\alpha_2$ and let x,y,z,u,x',y',z',u' be elements of $\underline{A}$. Then

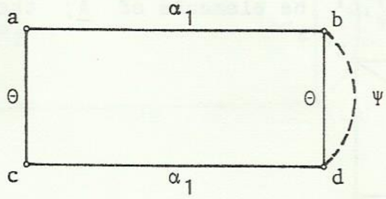


<u>implies</u>

<u>Proof:</u> Look at α_1 as a subalgebra of $\underline{A}\times\underline{A}$ and define congruences $\Theta\times\alpha_2$, $1\times\Theta$ and $\Psi\times\Theta$ on the algebra α_1 in the obvious way to obtain



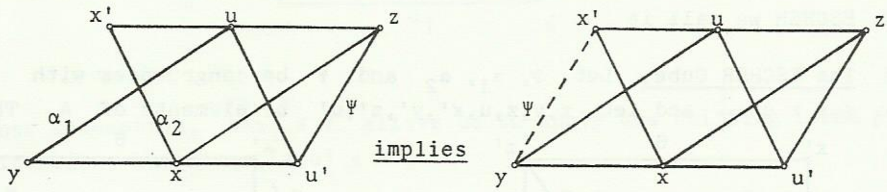
If the conditions for the Shifting Lemma are satisfied we get $(x',x)\ \psi\times\theta\ (y',y)$ and in particular $x'\ \psi\ y'$. Thus to show that $\theta\times\alpha_2 \wedge 1\times\theta \leq \psi\times\theta$ we take $((a,b),(c,d))$ from the left hand side and obtain:



To see that $(b,d) \in \psi$ we use $(b,d) \in \theta$ and $(b,d) \in \alpha_2$ and our hypothesis. Thus the latter Shifting Lemma provides for $(a,c) \in \psi$, i.e. $((a,b),(c,d))$ from the right hand side.

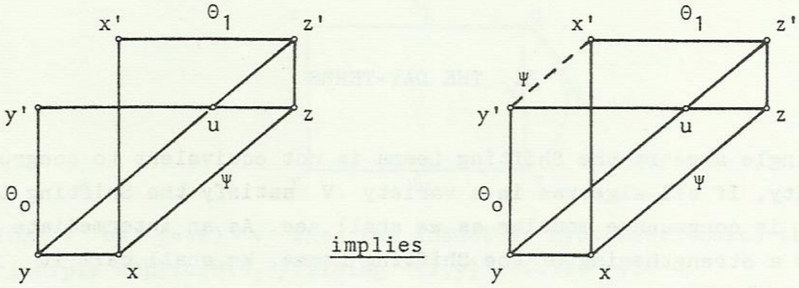
Again we are interested in two special cases. Firstly, on identifying z with z' and y with y' we obtain

2.6 <u>The Little Pappian Theorem:</u> If θ, α_1, α_2 and ψ are congruences with $\theta\wedge\alpha_1 \leq \psi \geq \theta\wedge\alpha_2$ and $x,y,z,u,x',u' \in \underline{A}$ then



Similarly, on identifying α_2 with ψ and u with u' we obtain

2.7 <u>Lemma:</u> Let θ_0, θ_1 and ψ be congruences with $\theta_0\wedge\theta_1 \leq \psi$ and x,y,z,u,x',y',z' elements of $\underline{A}$ then

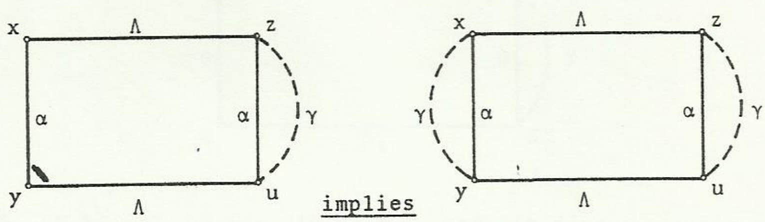


This lemma, as will turn out later is precisely what is needed to have groups in modular varieties being abelian.

3. THE DAY-TERMS

For a single algebra the Shifting Lemma is not equivalent to congruence modularity. If all algebras in a variety V satisfy the Shifting Lemma then V is congruence modular as we shall see. As an intermediate step we will use a strengthening of the Shifting Lemma. We shall call it

3.1 <u>The Shifting Principle:</u> Let α and γ be congruences and Λ a reflexive, symmetric and compatible relation on $\underline{A}$ with $(\alpha\cap\Lambda) \leq \gamma \leq \alpha$. For any elements $x,y,z,u \in \underline{A}$



The proof that the Shifting Principle holds in a modular variety will have to be postponed for a few pages. We shall first show that the Shifting Principle implies congruence modularity.

3.2 <u>Lemma:</u> If the Shifting Principle holds for any algebra $\underline{A}$ then $\underline{A}$ is congruence modular.

<u>Proof:</u> The proof is based on an idea of A. DAY [9]. Suppose $\alpha \geq \gamma$ and β are congruences. Then

$$\alpha \wedge (\beta \vee \gamma) = \bigcup_{n \in \mathbb{N}} \alpha \cap \Lambda_n$$

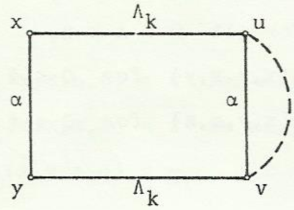
where

$$\Lambda_o := \beta \text{ and}$$
$$\Lambda_{n+1} := \Lambda_n \circ \gamma \circ \beta$$

Thus it suffices to show that

$$\alpha \cap \Lambda_n \subseteq (\alpha\wedge\beta) \vee \gamma \quad \text{for every } n \in \mathbb{N}.$$

For $n = 0$ this is trivial. Assuming then that $\alpha \cap \Lambda_k \subseteq (\alpha\wedge\beta) \vee \gamma$, $(x,y) \in \alpha \cap \Lambda_{k+1}$ implies that $(x,y) \in \alpha \cap (\Lambda_k \circ \gamma \circ \beta) \subseteq \alpha \cap (\Lambda_k \circ \gamma \circ \Lambda_k)$. Thus there exist $u,v \in \underline{A}$ with



Replacing γ by $(\alpha\wedge\beta)\vee\gamma$ then the inductive hypothesis makes the Shifting Principle applicable, yielding $(x,y) \in (\alpha\wedge\beta)\vee\gamma$.

To prepare an immediate application, let us define:

3.3 <u>Definition:</u> An algebra $\underline{A}$ has <u>regular congruences</u>, if every congruence is uniquely determined by any of its classes.

J. HAGEMANN used R. WILLE's Mal'cev-type characterization of regularity [41] to show that varieties of regular algebras are congruence modular (and even n-permutable for some n) [22]. Refining this theorem, S. BULMAN-FLEMING, A. DAY and W. TAYLOR proved [4];

3.4 <u>Theorem:</u> If all subalgebras of $\underline{A}\times\underline{A}$ have regular congruences then $\underline{A}$ is congruence modular.

<u>Proof:</u> We prove the Shifting Principle, namely, with notation as in 3.1 we take Λ any reflexive subalgebra of $\underline{A}\times\underline{A}$ and define congruences $\alpha\times\gamma$ and $\gamma\times\gamma$ on Λ by

$$(a,b)\ \alpha\times\gamma\ (c,d) \quad \text{iff} \quad a\ \alpha\ c \quad \text{and} \quad b\ \gamma\ d$$

and $$(a,b)\ \gamma\times\gamma\ (c,d) \quad \text{iff} \quad a\ \gamma\ c \quad \text{and} \quad b\ \gamma\ d.$$

Now for an arbitrary $a \in \underline{A}$ look at the (a,a)-class of $\alpha\times\gamma$. Namely, if $(a,a)\ \alpha\times\gamma\ (r,s)$ we get $(r,s) \in \alpha$ since $\gamma \leq \alpha$ and α is transitive. Hence $(r,s) \in \Lambda \cap \alpha$, therefore $(r,s) \in \gamma$, yielding $(a,a)\ \gamma\times\gamma\ (r,s)$ by transitivity of γ. We have just shown that $[(a,a)]\ \alpha\times\gamma = [(a,a)]\ \gamma\times\gamma$ and may now infer $\alpha\times\gamma = \gamma\times\gamma$ by regularity. Thus $(x,z)\ \gamma\times\gamma\ (y,u)$ in 3.1 hence $x\ \gamma\ y$.

Note that in fact we have shown slightly more, namely that <u>c-regularity</u> implies modularity. Here $\underline{c}$-regularity is a weakening of the notion of regularity. Algebras are supposed to have a constant $\underline{c}$, and every congruence is supposed to be determined by the class containing the constant $\underline{c}$.

We can now state and prove A. DAY's Mal'cev type characterization of congruence modular varieties [9]. The following chapter will very much depend on a deeper understanding of the geometrical meaning of those terms.

3.5 <u>Theorem:</u> A variety $\underline{V}$ is congruence modular if and only if for some natural number n there exist quaternary terms $m_o,\dots,m_n$ such that the following equations hold in $\underline{V}$:

(M0) $m_o(x,y,z,u) = x$

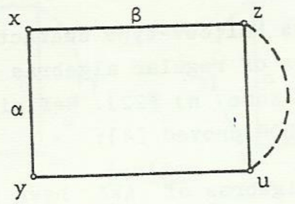
(M1) $m_i(x,x,y,y) = x$ for all $0 \le i \le n$,

(M2) $m_i(x,y,x,y) = m_{i+1}(x,y,x,y)$ for $0 \le i < n$, i even

(M3) $m_i(x,y,z,z) = m_{i+1}(x,y,z,z)$ for $0 \le i < n$, i odd

(M4) $m_n(x,y,z,u) = y$.

Proof: (existence):
Let $\underline{F}_{\underline{V}}(\{x,y,z,u\})$ be the free algebra in $\underline{V}$ freely generated by the set $X = \{x,y,z,u\}$. For $a,b \in X$ let $\Theta_{(a,b)}$ be the smallest congruence relation on $\underline{F}_{\underline{V}}(X)$ containing the pair (a,b). Thus for $\alpha := \Theta_{(x,y)} \vee \Theta_{(z,u)}$, $\beta := \Theta_{(x,z)} \vee \Theta_{(y,u)}$ and $\gamma := \Theta_{(z,u)}$ we set $\delta := (\alpha \wedge \beta) \vee \gamma$ to have the situation



The Shifting Lemma hence yields $(x,y) \in (\alpha \wedge \beta) \vee \gamma$ i.e. there exists a number n such that (x,y) is in the n-fold relational product of $\alpha \wedge \beta$ and γ. More precisely, for n there exist elements $m_o,\ldots,m_n$ of $\underline{F}_{\underline{V}}(X)$ such that the following relations hold:

(m0) $m_o = x$

(m2') $m_i (\Theta_{(x,y)} \vee \Theta_{(z,u)}) \wedge (\Theta_{(x,z)} \vee \Theta_{(y,u)}) m_{i+1}$ for i even, $0 \le i < n$

(m3') $m_i\ \Theta_{(z,u)} m_{i+1}$ for i odd, $0 \le i < n$

(m4) $m_n = y$.

(m2') together with (m3') may be replaced by

(m1) $m_i\ \Theta_{(x,y)} \vee \Theta_{(z,u)} x$ for all $0 \le i \le n$

(m2) $m_i\ \Theta_{(x,z)} \vee \Theta_{(y,u)} m_{i+1}$ for i even, $0 \le i < n$

(m3) $m_i\ \Theta_{(z,u)} m_{i+1}$ for i odd, $0 < i < n$.

Since every element of $\underline{F}_{\underline{V}}(X)$ may be written as a quaternary $\underline{V}$-term, and using MAL'CEV's argument [32] we obtain terms $m_o,\ldots,m_n$ satisfying the equations (M0),...,(M4) as they correspond to the relations (m0),...,(M4).

Sufficiency: We prove the Shifting Principle 3.1.
With the notation of 3.1 we define elements $\underline{m}_i := m_i(x,y,z,u)$ and

$\underline{m}_i' := m_i(x,y,x,y)$. Then the equations for the m_i ensure us of the relations

S1: $\underline{m}_i = m_i(x,y,z,u) \; \alpha \; m_i(x,x,z,z) = x$ for all i

S2: $\underline{m}_i' = m_i(x,y,x,y) \; \alpha \; m_i(x,x,x,x) = x$ as well as

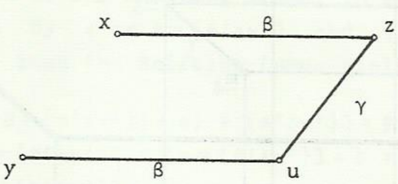
S3: $\underline{m}_i' = m_i(x,y,x,y) \wedge m_i(x,y,z,u) = \underline{m}_i$

S1, S2 and S3 jointly imply $\underline{m}_i' \; \gamma \; \underline{m}_i$ for all i.
For <u>i even</u> we have $\underline{m}_i' = \underline{m}_{i+1}'$, therefore with the above
S4: $\underline{m}_i \; \gamma \; \underline{m}_{i+1}$ for i even. For i odd the corresponding relation follows from
$\underline{m}_i = m_i(x,y,z,u) \; \gamma \; m_i(x,y,z,z) = m_{i+1}(x,y,z,z) \; \gamma \; m_{i+1}(x,y,z,u) = \underline{m}_{i+1}$.
Therefore with transitivity of γ we obtain
$x = \underline{m}_o \; \gamma \; \underline{m}_1 \; \gamma \; \underline{m}_2 \ldots \underline{m}_{n-1} \; \gamma \; \underline{m}_n = y$, i.e. $(x,y) \in \gamma$.
Now 3.2 completes the proof.

As a corollary we obtain

3.6 <u>Corollary:</u> In a variety of algebras the Shifting Lemma is equivalent to modularity, and both are equivalent to the Shifting Principle.

It is worthwhile, to look more closely at the points $\underline{m}_i$ and $\underline{m}_i'$ constructed in the above proof. Let therefore β and γ be congruences and x,y,z,u be points with



Under which conditions do these points form a β-γ-parallelogram, i.e. when do we have $x \equiv y (\text{mod } \gamma)$?
As above we will define again $\underline{m}_i := m_i(x,y,z,u)$. Then we get:

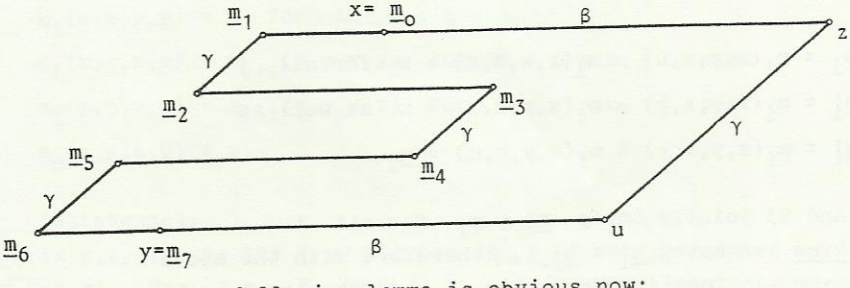
(a) $x = m_o$ by (M0)

(b) $\underline{m}_i \; \beta \; \underline{m}_{i+1}$ for i even by (M2)

(c) $\underline{m}_i \; \gamma \; \underline{m}_{i+1}$ for i odd by (M3)

(d) $\underline{m}_n = y$ by (M4)

This situation is shown in the following figure (using $n = 7$).

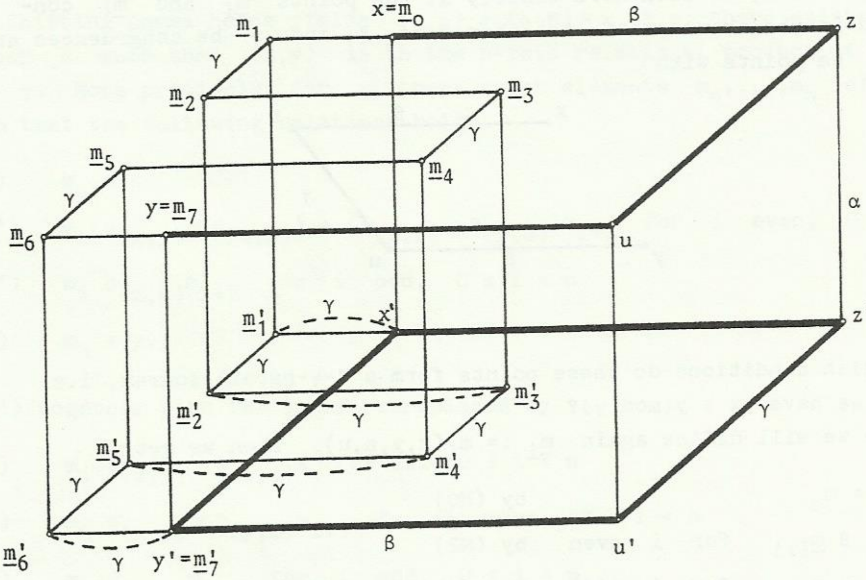


One direction of the following lemma is obvious now:

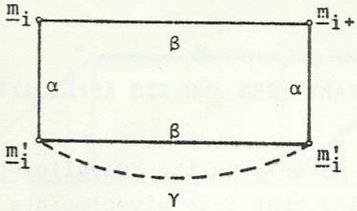
3.7 Lemma: $x \, \gamma \, y$ if and only if $\underline{m}_i \, \gamma \, \underline{m}_{i+1}$ for i even.

Notice that we had no chance so far, to use the powerful equations (M1). Only if we start with a completed parallelogram we can use (M1) and obtain $\underline{m}_i(x,y,z,u) \, \gamma \, \underline{m}_i(x,x,z,z) = x$. In particular $\underline{m}_i \, \gamma \, \underline{m}_{i+1}$ follows, finishing the proof of lemma 3.7.

Putting together a completed parallelogram and an uncompleted parallelogram, so that corresponding points are congruent modulo some congruence relation α, we obtain the following familiar figure:



Thus all we need for showing $x \, \gamma \, y$ is, to apply the Shifting Lemma to the situation



whenever i is even.

Thus all we need is the condition $\alpha \wedge \beta \leq \gamma$. This is precisely the original proof for the cube lemma.

Clearly we can easily formulate a theorem having as special cases all the configuration theorems of the preceding chapter. We call it the <u>Parallelogram Principle</u>:

3.8 <u>Parallelogram Principle:</u> To prove that (x,y,z,u) with $x\ \beta\ z\ \gamma\ u\ \beta\ y$ forms a γ-β-parallelogram, find a β'-γ'-parallelogram (x',y',z',u') such that (x,x'), (y,y'), (z,z') and (u,u') are from some congruence α. If $(\alpha \vee (\beta' \wedge \gamma')) \wedge \beta \leq \gamma$ holds then (x,y,z,u) is a γ-β-parallelogram.

Loosely spoken: "Look along some congruence α onto a completed parallelogram".

To prove it, just follow the reasoning after 3.7 with β and γ replaced by β' and γ' in the completed parallelogram. The only crucial step is the application of the Shifting lemma. In the picture of the preceding page replace β by $\phi := \beta \vee (\beta' \wedge \gamma')$ and γ by $\Psi := (\alpha \wedge \phi) \vee (\beta' \wedge \gamma')$. Thus the Shifting Lemma yields $\underline{m}'_i\ \phi \wedge \Psi\ \underline{m}'_{i+1}$ hence

$$
\begin{aligned}
(\underline{m}_i, \underline{m}_{i+1}) &\in [([\beta \vee (\beta' \wedge \gamma')] \wedge \alpha) \vee (\beta' \wedge \gamma')] \wedge \beta = \\
&= [\beta \vee (\beta' \wedge \gamma')] \wedge [\alpha \vee (\beta' \wedge \gamma')] \wedge \beta = \\
&= [\alpha \vee (\beta' \wedge \gamma')] \wedge \beta \leq \gamma .
\end{aligned}
$$

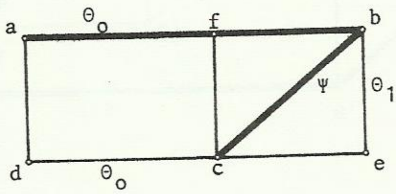
4. A SIXARY TERM AND ITS APPLICATIONS

In the last two chapters, notably in the "Parallelogram Principle" we developed a method for showing that four given points form a parallelogram. There is, however, no method visible to construct a fourth parallelogram point from three given ones. Clearly this is not possible in general, since it would imply congruence permutability. It turns out though, that, given some auxiliary points and congruences, parallelograms can be completed. In particular, parallelograms with one pair of sides being lines of a factor congruence can always be completed (Corollary 4.5). This chapter will in fact provide a term p, doing this uniformly throughout the variety.

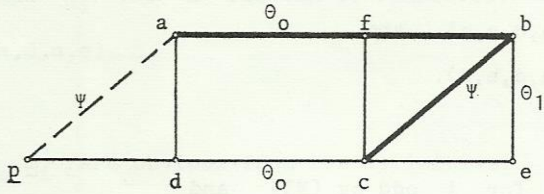
The construction of p relies almost exclusively on the geometrical visualization developed in former chapters and provides an excellent example of how geometry can inspire and lead algebraic computations. Thus instead of just writing down p and proving the characteristic relations, we will include the geometric reasoning which necessarily leads to the discovery of p.

The usefulness of p will become apparent in later chapters. In many instances this term simulates MAL'CEV's term from 1.2, thus allowing us to carry many results about permutable varieties from [16] over to modular varieties. In particular the methods of coordinatizing algebras in permutable varieties as developed in [16] need precisely Corollary 4.5 as an additional ingredient for being valid in modular varieties. This has been worked out in [17].

4.1 <u>Theorem:</u> In a modular variety $\underline{V}$ there exists a 6-ary term $p(x_1,\dots,x_6)$ with the following property: Let Θ_o, Θ_1 and Ψ be congruences on $\underline{A} \in \underline{V}$ with $\Theta_o \wedge \Theta_1 \leq \Psi$ and let a,b,c,d,e,f be elements of $\underline{A}$. Then the relations



imply that $\underline{p} := p(a,b,c,d,e,f)$ satisfies



This term p thus gives a fourth parallelogram point $\underline{p}$ for a,b and c, with d,e,f being auxiliary points.

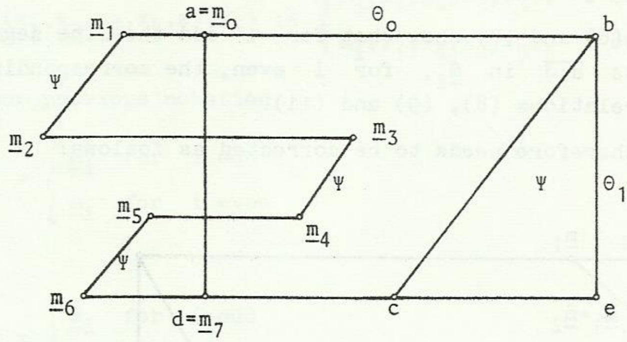
Let us then start with the hypothesis of the theorem. For the first part of the proof we can do without the point f. Since we also assumed that $\Theta_o \wedge \Theta_1 \leq \Psi$ we do not loose generality if we set $\Theta_o \wedge \Theta_1 = 0$. Firstly let us apply the DAY-terms to find the points $\underline{m}_i := m_i(a,d,b,c)$. Then the equations (M0), (M2), (M3) and (M4) yield the relations familiar from 3.7:

(1) $\underline{m}_o = a$

(2) $\underline{m}_i \; \Theta_o \; \underline{m}_{i+1}$ for i even

(3) $\underline{m}_i \; \Psi \; \underline{m}_{i+1}$ for i odd and

(4) $\underline{m}_n = d$.



Now geometrically we have to shift the Ψ-line $\overline{b,c}$ to join up with the point a so, that it intersects the Θ_o-line $\overline{d,e}$ in a point, which will be $\underline{p}$. This is not possible in just one step, hence we will proceed by shifting the line-segments $\overline{\underline{m}_i,\underline{m}_{i+1}}$ for i odd and then "add them together" to make them form the desired Ψ-line starting at a. The problem is that we have no control about as to where the $\overline{\underline{m}_i,\underline{m}_{i+1}}$ line segments are positioned "horizontally" (modulo Θ_o).

A first step in overcoming this difficulty is the observation that the segments $\overline{\underline{m}_i,\underline{m}_{i+1}}$ do have points of intersection with the Θ_1-line through a.

Namely define:

$\underline{\hat{m}}_i := m_i(a,d,c,c)$ and

$\underline{\tilde{m}}_i := m_i(a,d,b,e)$.

Then

(5) $\underline{\hat{m}}_i = \underline{\hat{m}}_{i+1}$ for i odd by (M3) and

(6) $\underline{\hat{m}}_i = \Theta_1\ a$ by (M1).

From the definition and using (5) we find

(7) $\underline{m}_i\ \Psi\ \underline{\hat{m}}_i\ \Psi\ \underline{m}_{i+1}$ for i odd.

Similarly for the $\underline{\tilde{m}}_i$ we find the relations

(8) $\underline{\tilde{m}}_i\ \Theta_o\ \underline{m}_i$ for every i by definition and

(9) $\underline{\tilde{m}}_i\ \Theta_1\ a$ by (M1).

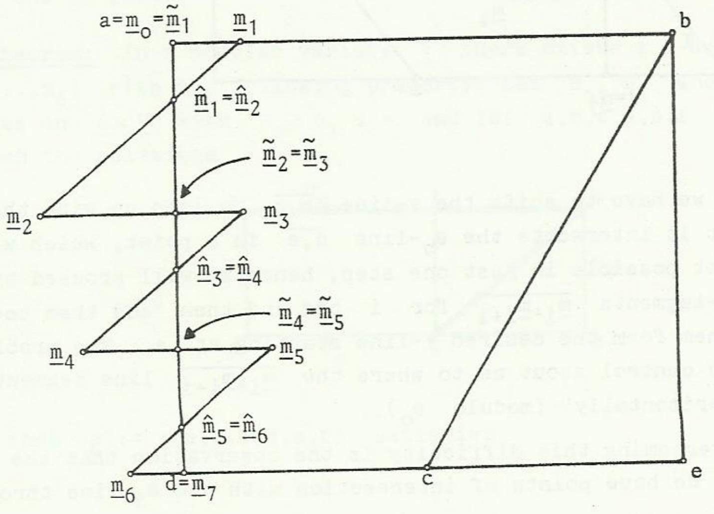
Using (9), (8) and (2) we conclude

(10) $\underline{\tilde{m}}_i\ \Theta_o \wedge \Theta_1\ \underline{\tilde{m}}_{i+1}$ if i is even, hence with our assumption:

(11) $\underline{\tilde{m}}_i = \underline{\tilde{m}}_{i+1}$ if i is even.

Now the relations (6) and (7) show that for i odd the line segment $\overline{\underline{m}_i,\underline{m}_{i+1}}$ intersects $\overline{a,d}$ in $\underline{\hat{m}}_i$, for i even, the corresponding fact follows from the relations (8), (9) and (11).

Our last picture therefore needs to be corrected as follows:



At this point we observe that the Θ_o-Ψ-parallelograms given by three points $\underline{\tilde{m}}_i$, $\underline{m}_i$ and $\underline{\hat{m}}_i$ for i odd can be completed. For this we define

$$\underline{\ddot{m}}_i := m_i(a,d,c,e).$$

The relations

(12) $\underline{\tilde{m}}_i \; \Psi \; \underline{\ddot{m}}_i \; \Theta_o \; \underline{\hat{m}}_i$ are obvious from the definition.

The points and relations collected so far provide a proof of the following lemma which is an important intermediate step in the proof of 4.1.

4.2 Lemma: There exist terms $s_o,\ldots,s_{n-1}$ and $t_1,\ldots,t_{n-1}$ in every modular variety such that with the hypothesis of 4.1 we have for $\underline{t}_i := t_i(a,b,c,d,e)$ and $\underline{s}_i := s_i(a,b,c,d,e)$ that $a = \underline{s}_o$, $d = \underline{s}_{n-1}$, and $\underline{s}_i \; \Psi \; \underline{t}_{i+1} \; \Theta_o \; \underline{s}_{i+1} \; \Theta_1 \; \underline{s}_i$ for every $i \le n-2$.

Proof: Define

$$t_i(x_1,x_2,x_3,x_4,x_5,x_6) := \begin{cases} m_i(x_1,x_4,x_3,x_5), & i \text{ odd} \\ m_i(x_1,x_4,x_2,x_3), & i \text{ even.} \end{cases}$$

and

$$s_i(x_1,x_2,x_3,x_4,x_5,x_6) := \begin{cases} m_i(x_1,x_4,x_3,x_3), & i \text{ odd} \\ m_i(x_1,x_4,x_2,x_5), & i \text{ even.} \end{cases}$$

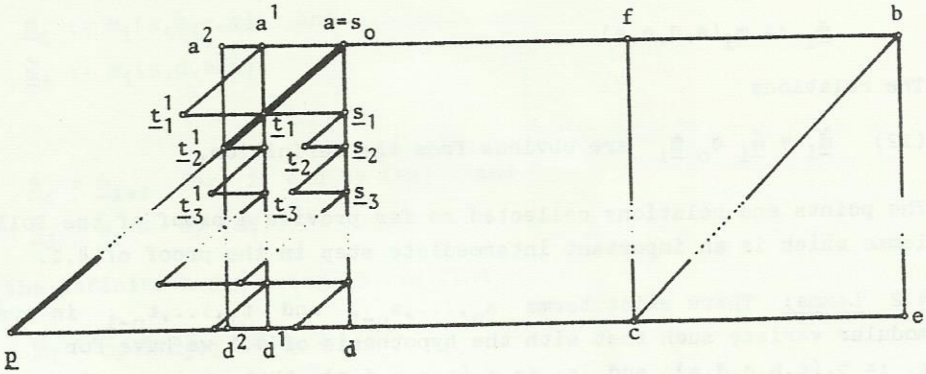
then in our previous notation

$$\underline{t}_i = \begin{cases} \underline{\ddot{m}}_i & \text{for } i \text{ odd} \\ \underline{m}_i & \text{for } i \text{ even} \end{cases}$$

and

$$\underline{s}_i = \begin{cases} \underline{\hat{m}}_i & \text{for } i \text{ odd} \\ \underline{\tilde{m}}_i & \text{for } i \text{ even.} \end{cases}$$

Now the obvious idea, motivated by the geometry is to apply Lemma 4.2 onto itself. This will become clear by looking at the picture on the next page. Applying Lemma 4.2 first yields us points $\underline{t}_1,\ldots,\underline{t}_{n-1}$ and $\underline{s}_o,\ldots,\underline{s}_{n-1}$. Then we shall construct new points a^1 and d^1 in the shown position. After having done that we shall apply Lemma 4.2 again but now replacing a by a^1 and d by d^1. This way another collection of points $\underline{t}_1^1, \underline{t}_2^1, \ldots, \underline{t}_{n-1}^1$ and $\underline{s}_o^1, \underline{s}_1^1, \ldots, \underline{s}_{n-1}^1$ is obtained. Now the choice of the points a^1 and d^1 will guarantee that $\underline{s}_1^1 = \underline{t}_1$. This way we have prolonged the short segment $\overline{a,\underline{t}_1}$ by the segment $\overline{\underline{t}_1,\underline{t}_2^1}$, which is just a parallel shift of $\overline{\underline{s}_1,\underline{t}_1}$. Thus continuing with the new points a^2 and d^2, and so on, we manage to connect all segments $\overline{\underline{s}_i,\underline{t}_{i+1}}$ to finally end up with the desired line $\underline{a},\underline{p}$.



We have promised to construct $\underline{p}$ as the value of a term. But our plan, presented above can easily be modified this way. Thus for the proof of 4.1 we define terms

$$a^o(x_1, \ldots, x_6) := x_1$$
$$b^o(x_1, \ldots, x_6) := x_2$$
$$\vdots \qquad \vdots$$
$$f^o(x_1, \ldots, x_6) := x_6$$

and $t_i^o(x_1, \ldots, x_6) := t_i(x_1, \ldots, x_6)$ as well as

$s_i^o(x_1, \ldots, x_6) := s_i(x_1, \ldots, x_6)$ where the t_i and s_i are the terms from Lemma 4.2. Further define recursively for $1 \le k \le n-1$:

$$a^k(x_1,\ldots,x_6) := t_k^{k-1}(x_1,x_2,x_6,x_1,x_2,x_6)$$
$$d^k(x_1,\ldots,x_6) := t_k^{k-1}(x_4,x_5,x_3,x_4,x_5,x_3)$$
$$t_i^k(x_1,\ldots,x_6) := t_i(a^k(x_1,\ldots,x_6),x_2,x_3,d^k(x_1,\ldots,x_6),x_5,x_6)$$
$$s_i^k(x_1,\ldots,x_6) := s_i(a^k(x_1,\ldots,x_6),x_2,x_3,d^k(x_1,\ldots,x_6),x_5,x_6)$$

Let us agree that with $\underline{a}^i$, $\underline{d}^i$, $\underline{s}_i^k$, $\underline{t}_i^k$ we mean the result from applying the term with the corresponding name onto the points a,b,c,d,e,f (in that order).

We need to show five relations:

(13) $\underline{a}^k \ \theta_1 \ \underline{d}^k$

(14) $\underline{a}^k \ \theta_o \ a$ and $\underline{d}^k \ \theta_o \ d$

(15) $\underline{t}_i^k \ \theta_o \ \underline{t}_i$ for $1 \le i \le n-1$

(16) $\underline{s}_i^k \ \theta_o \ \underline{s}_i$ for $i \le n-1$

(17) $\underline{t}_{k+1}^k \ \psi \ a$.

The first four relations are easy namely

$$\underline{a}^{k+1} = t^k_{k+1}(a,b,f,a,b,f)\ \Theta_1\ t^k_{k+1}(d,e,c,d,e,c) = \underline{d}^{k+1}$$

$$\underline{a}^{k+1} = t^k_{k+1}(a,b,f,a,b,f)\ \Theta_o\ t^k_{k+1}(a,a,a,a,a,a) = a$$

$$\underline{d}^{k+1} = t^k_{k+1}(d,e,c,d,e,c)\ \Theta_o\ t^k_{k+1}(d,d,d,d,d,d) = d$$

since all the terms m_i and hence all their composites are idempotent.
To prove (15) and (16) we need (14):

$\underline{t}^k_i = t_i(a^k,b,c,d^k,e,f)\ \Theta_o\ t_i(a,b,c,d,e,f) = \underline{t}_i$ and

$\underline{s}^k_i = s_i(a^k,b,c,d^k,e,f)\ \Theta_o\ s_i(a,b,c,d,e,f) = \underline{s}_i$.

Clearly (17) is the crucial relation. We use induction on k. The case $k=0$ comes from 4.2. In the induction step we first use (15), (16) and Lemma 4.2 to show that

(18) $\underline{t}^k_{k+1}\ \Theta_o\ \underline{t}_{k+1}\ \Theta_o\ \underline{s}_{k+1}\ \Theta_o\ \underline{s}^{k+1}_{k+1}$.

Now we apply 4.2 again with a replaced by $\underline{a}^{k+1}$ and d replaced by $\underline{d}^{k+1}$. Thus we obtain $\underline{s}^{k+1}_{k+1}\ \Theta_1\ \underline{a}^{k+1}$. On the other hand

$\underline{a}^{k+1} = t^k_{k+1}(a,b,f,a,b,f)\ \Theta_1\ t^k_{k+1}(a,b,c,d,e,f) = \underline{t}^k_{k+1}$. Thus

$\underline{s}^{k+1}_{k+1}\ \Theta_1\ \underline{t}^k_{k+1}$ and $\underline{s}^{k+1}_{k+1}\ \Theta_o\ \underline{t}^k_{k+1}$ from above, yielding

(19) $\underline{s}^{k+1}_{k+1} = \underline{t}^k_{k+1}$

according to our assumption that $\Theta_o \wedge \Theta_1 = 0$. Using the inductive hypothesis and Lemma 4.2 we finally arrive at

$\underline{t}^{k+1}_{k+2} = t_{k+2}(a^{k+1},b,c,d^{k+1},e,f)\ \Psi\ s_{k+1}(a^{k+1},b,c,d^{k+1},e,f) = \underline{s}^{k+1}_{k+1} = \underline{t}^k_{k+1}\ \Psi\ a$,

proving (17).

Hence defining $p(x_1,\dots,x_6) := t^{n-2}_{n-1}(x_1,\dots,x_6)$ we combine (15) and (17) to have for $k = n-2$:

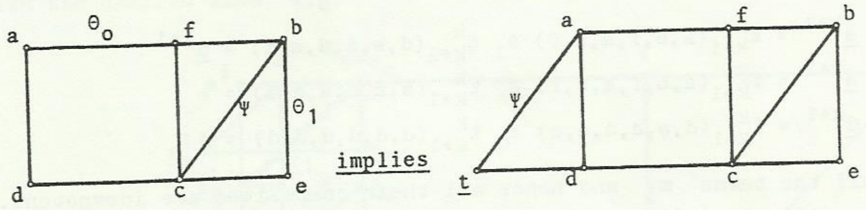
$$p(a,b,c,d,e,f) = \underline{t}^{n-2}_{n-1}\ \Theta_o\ \underline{t}_{n-1}\ \Theta_o\ d \quad \text{and}$$

$$p(a,b,c,d,e,f) = \underline{t}^{n-2}_{n-1}\ \Psi\ a.$$

This finishes the proof of 4.1.

We can use the techniques from the above proof to make the following improvement:

4.3 <u>Theorem:</u> There exists a ternary term $t(x,y,z)$ in every modular variety such that given congruences Θ_o, Θ_1 and Ψ with $\Theta_o \wedge \Theta_1 \leq \Psi$ and elements a,b,c,d,e,f with



with $\underline{t} := t(d,e,c)$. Moreover, the equation $t(x,y,y) = x$ holds in $\underline{V}$.

Proof: Define $t(x,y,z) := p(x,y,z,x,y,z)$, then
$t(d,e,c) = p(d,e,c,d,e,c)\ \Theta_1\ p(a,b,c,d,e,f)$ and
$t(d,e,c)\ \Theta_o\ t(d,d,d) = d\ \Theta_o\ p(a,b,c,d,e,f)$.
Thus $t(d,e,c)\ \Theta_o \wedge \Theta_1\ p(a,b,c,d,e,f)$.
To see that the equation $t(x,y,y) = x$ holds in $\underline{V}$, take an algebra $\underline{A} \in \underline{V}$ and $x,y \in \underline{A}$. Consider the direct product $\underline{A} \times \underline{A}$ and set $\Theta_o := \pi_1$, $\Theta_1 := \pi_2$ and $\Psi := \pi_1$. With $a = (x,y)$, $b = f = (y,y)$, $e = c = (y,x)$ and $d = (x,x)$ the geometric conditions demand that $t((x,x),(y,x),(y,x))$ has to be (x,x). Evaluating the first component gives us $t(x,y,y) = x$.

4.4 Corollary: Let Θ_o, Θ_1 and Ψ be congruences with $\Theta_o \wedge \Theta_1 \leq \Psi \leq \Theta_o \vee \Theta_1$. If Θ_o permutes with Θ_1 then Ψ permutes with Θ_o and with Θ_1.

Proof: Suppose $a\ \Theta_o\ b\ \Psi\ c$ then $(b,c) \in \Theta_o \circ \Theta_1$, $(b,c) \in \Theta_1 \circ \Theta_o$ and $(a,c) \in \Theta_o \circ \Psi \subseteq \Theta_o \circ \Theta_o \circ \Theta_1 \subseteq \Theta_o \circ \Theta_1 \subseteq \Theta_1 \circ \Theta_o$ imply the existence of further points d,e,f with the configuration of 4.3. Applying 4.3 therefore we get $a\ \Psi\ p(d,e,c)\ \Theta_o\ c$.

The form in which this corollary usually will be applied is as follows:

4.5 Corollary: Let $\underline{B} = \prod_{i \in I} \underline{A}_i$ be the direct product of the algebras $\underline{A}_i$, i I. Then every congruence relation on $\underline{B}$ permutes with every factor congruence.

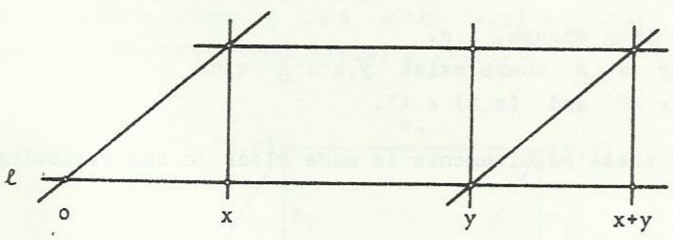
This corollary has many useful applications in GUMM, HERRMANN [21] where it guarantees that certain lattice theoretical decompositions (in the congruence lattices) actually yield algebraic decompositions.

More applications are provided in the following chapter. The following corollary appears in WERNER [39] under the assumption that $\underline{B}$ has 3-permutable congruences:

4.6 Corollary: Let $\underline{B} = \prod_{i \in I} \underline{A}_i$ be the direct product of the algebras $\underline{A}_i$, $i \in I$. For a subset $S \subseteq I$ let π_S be the canonical projection onto $\underline{C} := \prod_{i \in S} \underline{A}_i$. For any congruence Θ on $\underline{B}$, the image of Θ under π_S (= $\{(\pi_S(a),\pi_S(b)) \mid (a,b) \in \Theta\}$) is a congruence relation on $\underline{C}$.

5. COORDINATIZATION

To coordinatize a line ℓ in Desarguesian affine geometry we would embed ℓ in a plane and choose two more lines, ℓ' and ℓ'' to obtain a set of three mutually nonparallel lines. After choosing an element o from ℓ arbitrarily, two points x and y on ℓ are added according to the following figure



where horizontal, vertical and skew lines are lines parallel to ℓ, ℓ' and ℓ''.

This construction is known to yield an abelian group $\underline{G} = (\ell,+,o)$. Moreover the particular choice of o is irrelevant because, treating o as a variable, we actually have defined the ternary operation $x-o+y$.

In this chapter we will do precisely the same thing as above with an algebra $\underline{A}$ in a modular variety $\underline{V}$ playing the rôle of the line ℓ. The details we know about the congruence class geometry will provide us with the necessary tools making this process work. In particular, we will assume throughout this chapter that the algebra $\underline{A}$ is contained in a modular variety $\underline{V}$.

Thus we let the algebra $\underline{A}$ play the rôle of the line ℓ. Naturally our "plane" will be $\underline{A}\times\underline{A}$. There are two canonical congruences π_1 and π_2 on $\underline{A}\times\underline{A}$. Let $\ell := [(a,b)]\pi_2 = \{(x,b) \mid x \in \underline{A}\}$ be a class of π_2 for some $(a,b) \in \underline{A}\times\underline{A}$ chosen at will. Clearly A may be identified with the points of ℓ. For our line ℓ' an obvious candidate is found in $[(a,b)]\pi_1 = \{(a,x) \mid x \in A\}$. Now for ℓ'' we would like to choose the "diagonal", i.e. $\mathrm{diag}(\underline{A}) := \{(x,x) \mid x \in A\}$. Unfortunately $\mathrm{diag}(\underline{A})$ need not be a "line" in our sense, because lines are congruence classes. Thus we are forced to consider the smallest line containing $\mathrm{diag}(\underline{A})$, i.e. we set

$$\Delta := \langle\{((x,x),(y,y)) \mid (x,y) \in \underline{A}\times\underline{A}\}\rangle_{\underline{A}\times\underline{A}}.$$

Then ℓ'' has to be $[(x,x)]\Delta$ for some (any) x from $\underline{A}$.

We will go through this chapter however, assuming that ℓ'' is a line. This assumption is equivalent to: ℓ'' intersects ℓ and ℓ' in at most one point (see 5.1 below).

At this stage it is not clear whether we loose any generality by taking this particular choice for ℓ''. Of course for a "third line" ℓ'', in order to qualify for the geometric construction explained above two requirements are essential:

(L1) ℓ'' intersects ℓ and ℓ' in at most one point, i.e. if (x,y) and (x,z) are on ℓ'' then $y = z$ and symmetricall if (y,x) and (z,x) are on ℓ'' then $y = z$.

(L2) ℓ'' is long enough, i.e. for any $x \in \underline{A}$ there exist $y,z \in \underline{A}$ with $(x,y) \in \ell''$ and $(z,x) \in \ell''$.

The status of these requirements is made clear in the following observation:

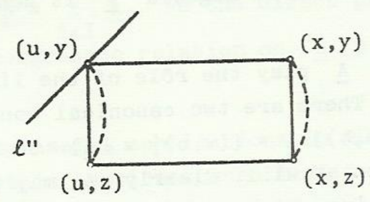
5.1 <u>Lemma:</u> Let Θ be a congruence relation on $\underline{A} \times \underline{A}$. Then the following are equivalent:

(i) Some class of Θ is a line satisfying (L1) and (L2).

(ii) Every class of Θ is a line satisfying (L1) and (L2).

(iii) Θ is a common complement to the factor congruences on $\underline{A} \times \underline{A}$, i.e. $\Theta \vee \pi_1 = \Theta \vee \pi_2 = 1_{\underline{A}\times\underline{A}}$ and $\Theta \wedge \pi_1 = \Theta \wedge \pi_2 = 0_{\underline{A}\times\underline{A}}$.

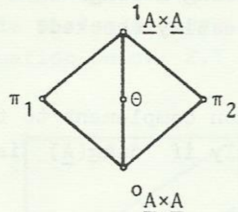
<u>Proof:</u> (i) $\to$ (iii): $\Theta \vee \pi_1 = \Theta \vee \pi_2 = 1_{\underline{A}\times\underline{A}}$ comes from (L2). Suppose $\Theta \wedge \pi_1 \neq 0_{\underline{A}\times\underline{A}}$, then for some $x,y,z \in \underline{A}$ with $y \neq z$ we have $(x,y)\ \Theta\ (x,z)$. Choose $u \in A$ with $(u,z) \in \ell''$ then (u,z) and (u,y) are both on ℓ'' by an application of the Shifting Lemma.



This contradiction proves (iii).

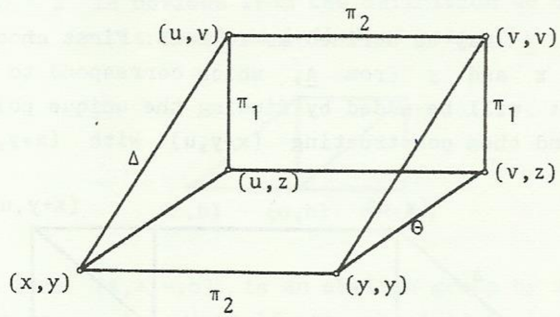
(iii) $\to$ (ii): (L1) is trivial for every class of Θ. Let ℓ be a class of Θ, $x \in A$ and $(a,b) \in \ell$. Then $(x,x)\ \pi_1 \vee \Theta\ (a,b)$ and, since Θ permutes with π_1 there exists a y from A with $(x,x)\ \pi_1\ (x,y)\ \Theta\ (a,b)$, hence $(x,y) \in \ell$.

In other words, Θ together with π_1 and π_2 form a 0-1-sublattice (named $\underline{M}_3$) of $\mathrm{Con}(\underline{A}\times\underline{A})$ as in the following lattice diagram:



Next we show that we might as well assume that diag($\underline{A}$) is a class of some congruence Δ. How to define Δ?

Suppose $(x,y)\ \Delta\ (u,v)$. Then certainly we want $(y,y)\ \Delta\ (v,v)$. By the properties of θ there exists a $z \in \underline{A}$ with $(x,y)\ \theta\ (u,z)$. Now the little Desarguesian Theorem forces $(y,y)\ \theta\ (v,z)$.



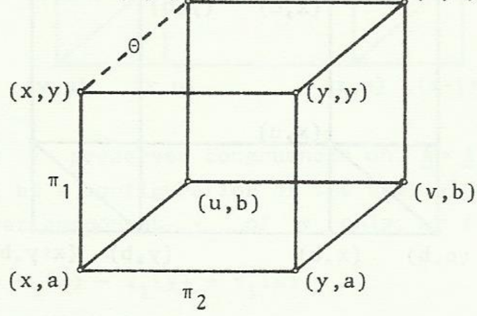
This reasoning goes back and forth between θ and Δ thus we are forced to define:

$$(x,y)\ \Delta\ (u,v) \quad :\underline{\text{iff}}\quad \exists z \in A \quad (x,y)\ \theta\ (u,z) \text{ and } (y,y)\ \theta\ (v,z).$$

We claim that this definition is equivalent to

$$(x,y)\ \Delta\ (u,v) \quad :\underline{\text{iff}}\quad \exists a,b \in A \quad (x,a)\ \theta\ (u,b) \text{ and } (y,a)\ \theta\ (v,b).$$

Namely, if the right side is true then, since $\pi_1 \circ \theta = 1_{\underline{A}\times\underline{A}}$ there exists a $z \in A$ with $(y,y)\ \theta\ (v,z)$ hence the Cube Lemma yields: $(x,y)\ \theta\ (u,z)$.



Now the properties of Δ, being a congruence relation on $\underline{A} \times \underline{A}$ with diag($\underline{A}$) a class of Δ are easily checked.

Thus we obtain:

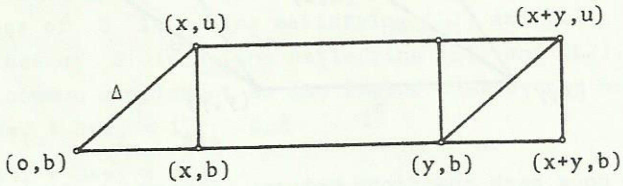
5.2 Lemma: There is a common complement to the factor congruences π_1 and π_2 on $\underline{A} \times \underline{A}$ if and only if diag($\underline{A}$) is a class of some congruence on $\underline{A} \times \underline{A}$.

From now on we may assume diag($\underline{A}$) being a line. Geometrically we know then:

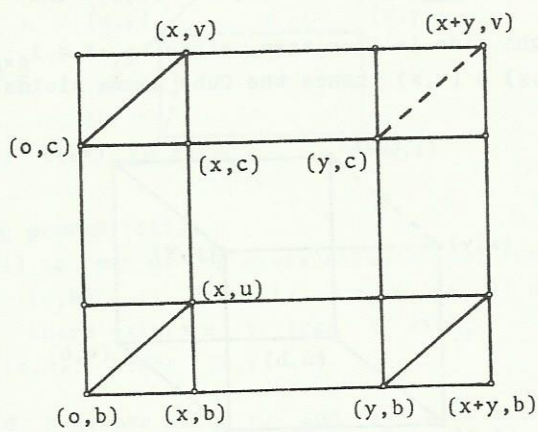
(11) any two lines (parallel to ℓ, ℓ' or ℓ'') intersect in at most one point and

(12) any two nonparallel lines (of the above) intersect in at least one point.

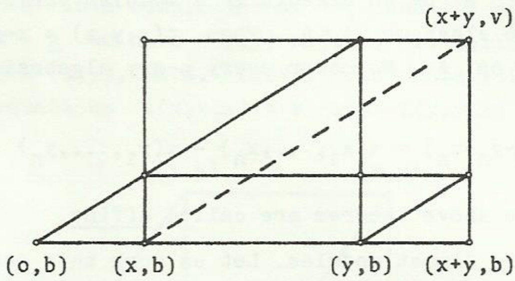
Thus addition on $\underline{A}$ may be defined as follows: First choose some element $o \in \underline{A}$. Elements x and y from $\underline{A}$, which correspond to points (x,b) and (y,b) on ℓ will be added by finding the unique point (x,u) with $(o,b)\ \Delta\ (x,u)$ and then constructing $(x+y,u)$ with $(x+y,u)\ \Delta\ (y,b)$.



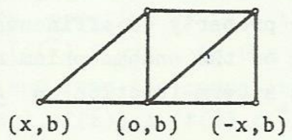
(11) and (12) from above guarantee that this process works and defines a binary operation $+$ on $\underline{A}$. First we show independence of the choice of b. Thus suppose we had used $c \in \underline{A}$ instead of b. Clearly we are done if we can show $(y,c)\ \Delta\ (x+y,v)$ in the picture below.



So the Cube Lemma gives the needed result. It is involved again in showing associativity of $+$. This is a simple exercise. For commutativity we need $(x+y,v)\ \Delta\ (x,b)$ in the situation below. 2.7 is all we need here:



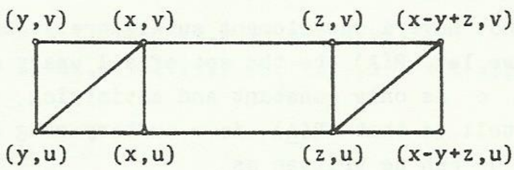
Since $o+x = x+o = x$ is obvious from the definition we only have to find $-x$. But, given x, $-x$ is found as indicated below:



Clearly we have that $(A,+,-,o)$ is an abelian group by now. Had we chosen a different element $e \in A$, we would have obtained an isomorphic group $(A,\underset{e}{+},\underset{e}{-},e)$ with the isomorphism defined by $x \to x+e$.

The arbitrariness of the choice of the neutral element is removed if we consider the ternary operation $x-y+z$. Comparing the construction of $x-y+z$ with Theorem 4.3, we find that it agrees with our ternary term $t(x,y,z)$. In other words, $x-y+z$ is given as a term function on $\underline{A}$.

Let us now see how the other operations of $\underline{A}$ behave with respect to $x-y+z$. To this end consider an algebraic function τ on $\underline{A} \times \underline{A}$. Since $x-y+z = w$ if and only if for some elements u and v the configuration



is given and since τ preserves congruences on $\underline{A} \times \underline{A}$, the image of this configuration will be a configuration of the same kind. In particular, looking at the first component τ_1 of τ only, we find the equation

$$\tau_1(x-y+z) = \tau_1(x) - \tau_1(y) + \tau_1(z).$$

Clearly this extends to τ being n-ary.

Let us collect the results achieved so far in a theorem:

5.3 Theorem: Let $\underline{A}$ be an algebra in a modular variety with diag($\underline{A}$) being a congruence class on $\underline{A} \times \underline{A}$. Then $t(x,y,z) = x-y+z$ for some abelian group defined on $\underline{A}$. Moreover every n-ary algebraic function τ on $\underline{A}$ satisfies:

$$\tau(x_1-y_1+z_1,\ldots,x_n-y_n+z_n) = \tau(x_1,\ldots,x_n) - \tau(y_1,\ldots,y_n) + \tau(z_1,\ldots,z_n).$$

Algebras as in the above theorem are called affine.

Affine algebras are almost modules. Let us work this out now. Again we look at a special case first. Suppose that $\underline{A}$ has a one-element subalgebra {o}. We then let o be the neutral element of our abelian group defined on $\underline{A}$ by $x+y := x-o+y$. Let R be the set of all unary algebraic functions τ on $\underline{A}$ which have no other constant than o in their representation. (If o is given by some constant, R is just the set of all unary term functions). The property of affineness clearly states that $\underline{R}$ can be viewed as a subring of the endomorphism ring of (A,+,-,o). Moreover, if $f(x_1,\ldots,x_n)$ is a term-function on $\underline{A}$ we may write

$$\begin{aligned} f(x_1,\ldots,x_n) &= f(x_1-o+o,o-o+x_2,\ldots,o-o+x_n) = \\ &= f(x_1,o,\ldots,o) + f(o,x_2,\ldots,x_n) = \\ &= \ldots \\ &= f(x_1,o,\ldots,o) + f(o,x_2,o,\ldots,o) + \ldots + f(o,\ldots,o,x_n) \\ &= \tau_1(x_1) + \tau_2(x_2) + \ldots + \tau_n(x_n) \\ &= \sum_{i=1}^{n} \tau_i x_i \end{aligned}$$

where product is taken in the ring $\underline{R}$.

Thus there is a module structure defined on $\underline{A}$ such that every operation on A is linear. Moreover, if o is an algebraic constant then the linear operations are precisely the term-functions of $\underline{A}$.

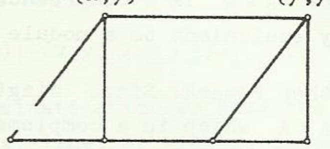
Now if $\underline{A}$ does not have a one-element subalgebra, then, after choosing an element o, we let R($\underline{A}$) be the set of all unary algebraic functions τ on $\underline{A}$ having o as only constant and satisfying $\tau(o) = o$. The above arguments again tell us that R($\underline{A}$) is a unitary ring and every term function $f(x_1,\ldots,x_n)$ can be written as

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} \tau_i x_i + a,$$

where a is a fixed element of $\underline{A}$, depending only on o and f, in fact $a = f(o,\ldots,o)$ and $\tau_i(x) = f(o,\ldots,o,x,o,\ldots,o) - f(o,\ldots,o)$.

Thus, if $\underline{A}$ is affine, then $\underline{A}$ is polynomially equivalent, i.e. has the same polynomial functions as the module ${}_{R(\underline{A})}A$.

There is another way to obtain the abelian group structure on $\underline{A}$. Suppose again that $\mathrm{diag}(\underline{A})$ is a congruence class for a congruence relation Δ on $\underline{A}\times\underline{A}$. From 5.1 Δ is a complement of π_1 and of π_2. Moreover, since for any $x,y,z \in \underline{A}$ we have $(y,y)\ \Delta\ (z,z)$ it follows that $t((x,z),(y,z),(z,z))\ \Delta\ (x,y)$, i.e. $(t(x,y,z),z)\ \Delta\ (x,y)$. From the properties of Δ the equations $t(x,x,z) = z$ and $t(y,y,z) = z$ follow



Hence t is a Mal'cev-term (as in 1.2), so $\underline{A}$ generates a permutable variety. Let f be any n-ary operation of $\underline{A}$, then in particular $(t(f(\vec{x}),f(\vec{y}),f(\vec{z})),\ f(\vec{z}))\ \Delta\ (f(\vec{x}),f(\vec{y}))$. On the other hand, since $(t(x_i,y_i,z_i),z_i)\ \Delta\ (x_i,y_i)$ and from the compatibility of Δ we conclude $(f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n)),\ f(\vec{z}))\ \Delta\ (f(\vec{x}),f(\vec{y}))$. Since the right sides are equal, the left sides are congruent. Since their second components coincide, so do their first components and we find: $t(f(\vec{x}),f(\vec{y}),f(\vec{z})) = f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n))$.

5.4 Proposition: An algebra $\underline{A}$ is affine if and only if there is a terrary term $t(x,y,z)$ on $\underline{A}$ satisfying $t(x,y,y) = x$, $t(x,x,z) = z$ and if every operation f of $\underline{A}$ sommutes with t.

Proof: t commutes with every operation means that $t\colon \underline{A}^3 \to \underline{A}$ is a homomorphism, so t commutes with every term function of $\underline{A}$ as well. In particular it commutes with itself, i.e.

a) $t(x,y,z) = t(t(x,y,y),t(x,x,y),t(z,x,x))$
$\quad = t(t(x,x,z),t(y,x,x),t(y,y,x)) = t(z,y,x)$

b) $t(t(x,y,z),y,v) = t(t(x,y,z),t(y,y,y),t(y,y,v))$
$\quad = t(t(x,y,y),t(y,y,y),t(z,y,v)) = t(x,y,t(z,y,v))$

and similarly

c) $t(x,y,t(y,x,y)) = y$.

Choosing an arbitrary element $o \in \underline{A}$ and setting $y = o$ in the above equations we find that $x+y := t(x,o,y)$ defines an abelian group structure on $\underline{A}$ with $t(x,y,z) = x-y+z$ and neutral element o. The rest follows with 5.3.

Let us keep a record now of the several characterizations of affine algebras. (More of them will result from the next chapters.)

5.5 Theorem: Let $\underline{A}$ be an algebra in a modular variety, then the following are equivalent:

(i) $\underline{A}$ is affine.

(ii) $\underline{A}$ has a Mal'cev-term t and every operation of $\underline{A}$ commutes with t.

(iii) There is a congruence relation Θ on $\underline{A} \times \underline{A}$ which is a common complement of π_1 and of π_2.

(iv) diag($\underline{A}$) is a congruence class on $\underline{A} \times \underline{A}$.

(v) Every subalgebra of $\underline{A} \times \underline{A}$ is a congruence class.

(vi) $\underline{A}$ is polynomially equivalent to a module over a ring with unit.

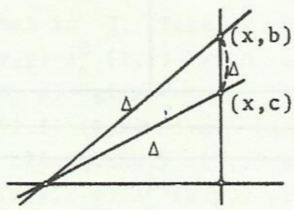
Theorem 5.3 deserves another remark: Since diag($\underline{A}$) is a congruence class of a congruence Δ on $\underline{A} \times \underline{A}$ which is a complement of π_1 and of π_2, and since all those congruences permute, we find that $\underline{A} \times \underline{A} \cong \underline{A} \times (\underline{A} \times \underline{A})/_\Delta$. Since a subalgebra of $\underline{A} \times \underline{A}$, namely diag($\underline{A}$) is identified by Δ, $(\underline{A} \times \underline{A})/_\Delta$ has a one-element subalgebra whilst this need not be so for $\underline{A}$. In particular, $\underline{A}$ and $\underline{A}/_\Delta$ need not be isomorphic. $\underline{A}/_\Delta$ can also be defined by changing each operation f into a new operation f^∇ by $f^\nabla(x_1,\dots,x_n) := f(x_1,\dots,x_n) - f(o,\dots,o)$ yielding the "linearization" $\underline{A}^\nabla$ of $\underline{A}$. This situation is examined more closely and in greater generality in section 9.

6. COMMUTATORS

So far we have assumed that diag($\underline{A}$) be a congruence class on $\underline{A} \times \underline{A}$. Suppose now this was not the case. We would then consider the smallest congruence relation Δ such that diag($\underline{A}$) is contained in a Δ-class. Thus we must set

$$\Delta := \langle\{((x,x),(y,y)) \mid (x,y) \in \underline{A} \times \underline{A}\}\rangle_{\underline{A}\times\underline{A}} .$$

If we do now try our coordinatization we have the difficulty that intersections of Δ-lines with π_1-lines or with π_2-lines need not be unique, hence the construction of $x-y+z$ is not unique.



Thus we have $(x,b)\ \Delta\ (x,c)$ <u>for some</u> $x \in \underline{A}$. Fortunately the Shifting Lemma tells us that in this case

$$(y,b)\ \Delta\ (y,c) \quad \underline{\text{for every}} \quad y \in \underline{A}.$$

Thus the following is a congruence relation on $\underline{A}$:

$$[1,1] := \{(b,c) \mid (x,b)\ \Delta\ (x,c) \quad \text{for some} \quad x \in A\}$$

and it is equal to

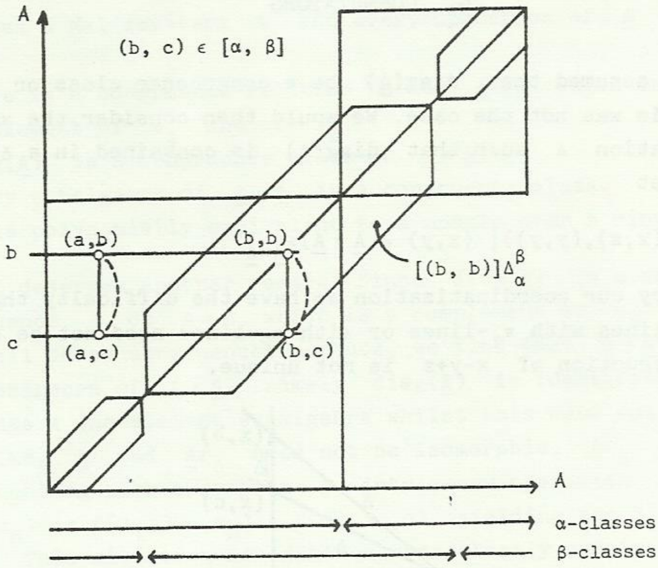
$$\{(b,c) \mid (b,b)\ \Delta\ (b,a)\}.$$

Factoring by this congruence relation we will indeed get an algebra $B = \underline{A}/[1,1]$ where now diag($\underline{B}$) is a congruence class in $\underline{B} \times \underline{B}$ and hence $\underline{B}$ is polynomially equivalent to a module. This will become clear later in this chapter in a more general setting.

Let us now generalize the above notions to obtain the important definition of a <u>commutator</u> of congruences. Let α and β be congruences on the algebra $\underline{A}$. We think of α now as a subalgebra of $\underline{A} \times \underline{A}$ and define a congruence relation on α by

$$\Delta_\alpha^\beta := \langle\{((x,x),(y,y)) \mid (x,y) \in \beta\}\rangle_\alpha .$$

Then $[\alpha,\beta] := \{(b,c) \mid (b,b)\ \Delta_\alpha^\beta\ (b,c)\}$ is called the commutator of α and β.



Just as $[1,1]$ measures the "thickness" of the smallest line containing diag($\underline{A}$), the commutator $[\alpha,\beta]$ measures the "thickness" of the diagonal pieces given by β in the algebra α. This is shown in the preceding figure where the two squares represent two blocks of α (as subsets of $\underline{A}\times\underline{A}$!).

Commutators were introduced into General Algebra by J.D.H. SMITH in [36]. He used them as a major tool for studying permutable varieties (which he calls Mal'cev varieties). His fundamental concept is "centrality". Here two congruences centralize each other if their commutator is 0. Starting with this concept he develops commutators. We believe that his approach is less direct and harder to work with than the approach we suggest.

J. HAGEMANN and C. HERRMANN [24] managed to carry the whole concept over to modular varieties. They analyized the lattice theoretical properties of Con(α) that made SMITH's concept work and gave a new definition of the commutator in modular varieties which coincided with SMITH's concept in the permutable case. Although many results true in permutable varieties could be proved again in modular varieties and an impressing list of new results could be added, the simplicity and clarity of the concept is lost and consequently their methods are unusually difficult to comprehed and to work with.

In our approach, the vehicle for transferring the concept from permutable to modular varieties is, of course, geometry.

Let us first see what this notion amounts to in some familiar varieties as in groups, rings (not necessarily associative), and lattices.

To facilitate the computations, we will freely use a result which we are going to prove later, namely that $[\alpha,\beta] \le \gamma$ if and only if $[\vec{\phi}\alpha,\vec{\phi}\beta] = 0$, where ϕ is the canonical homomorphism from $\underline{A}$ onto $\underline{A}/\gamma$, see 6.17 below.

6.1 Groups: Via the obvious translation between congruences and normal subgroups given by

$$x\ \Theta\ y \iff x\cdot y^{-1} \in N(\Theta) \quad \text{and}$$

$$x \in N \iff x\ \Theta(N)\ 1$$

we claim that the above definition captures precisely the notion of commutator (normal) subgroup of two normal subgroups.

Thus let $\underline{N}$ and $\underline{M}$ be normal in $\underline{G}$. Take $x \in N$, $y \in M$. Then $(y,1) \in \Theta(M) =: \beta$ hence $(y,y)\ \Delta_\alpha^\beta\ (1,1)$ with $\alpha := \Theta(N)$. Multiplication by $(x,1)$ and $(1,x)$ (from α) gives $(x,1)\cdot(y,y)\cdot(1,x)\ \Delta_\alpha^\beta\ (x,1)\cdot(1,1)\cdot(1,x)$ i.e. $(x\cdot y,y\cdot x)\ \Delta_\alpha^\beta\ (x,x)$. The fact that $(x,x\cdot y) \in \beta$ and consequently $(x,x)\ \Delta_\alpha^\beta\ (x\cdot y,x\cdot y)$ together with transitivity give us $(x\cdot y,x\cdot y)\ \Delta_\alpha^\beta\ (x\cdot y,y\cdot x)$, showing that $x\cdot y\ [\alpha,\beta]\ y\cdot x$. Therefore the above commutator contains the usual group theoretic commutator.

Equality is seen using 6.17. We factor our given group $\underline{G}$ by the group theoretic commutator $[N,M]$. ($\gamma = \Theta[N,M]$ in 6.17). Hence we may assume that N centralizes M, i.e. every $x \in N$ commutes with every $y \in M$. It follows that $N_\alpha^\beta := \{(x,x) \mid x \in M\}$ is a normal subgroup of α. Hence defining a congruence relation Θ_α^β on α by

$$(a,b)\ \Theta_\alpha^\beta\ (c,d) \ :\iff (a\cdot c^{-1},b\cdot d^{-1}) \in N_\alpha^\beta, \text{ i.e.}$$

$$\iff a\cdot c^{-1} = b\cdot d^{-1} \text{ and } a\cdot c^{-1} \in M$$

yields $\Theta_\alpha^\beta \ge \Delta_\alpha^\beta$.

Hence $(a,a)\ \Delta_\alpha^\beta\ (a,b) \Rightarrow (a,a)\ \Theta_\alpha^\beta\ (a,b) \Rightarrow (1,a\cdot b^{-1}) \in N_\alpha^\beta \Rightarrow a = b$. Thus $[\alpha,\beta] = 0$ finishing the claim.

6.2 Rings: For ideals I,J we get $[I,J] = (I\cdot J + J\cdot I)$, i.e. the ideal generated by all sums $i\cdot j + j\cdot i$ with $i \in I$ and $j \in J$. The proof is analogous to the preceding one.

Firstly for $i \in I$ and $j \in J$ we get with $\alpha = \Theta(I)$ and $\beta = \Theta(J)$ that $(j,j)\ \Delta_\alpha^\beta\ (o,o)$, hence $(j,j)\cdot(i,o)\ \Delta_\alpha^\beta\ (o,o)\cdot(i,o)$ whence $(j\cdot i,o)\ \Delta_\alpha^\beta\ (o,o)$. This proves $J\cdot I \subseteq [I,J]$. Similarly $I\cdot J \subseteq [I,J]$, therefore $(I\cdot J + J\cdot I) \subseteq [I,J]$.

For the reserve inclusion 6.17 again permits us to assume that $(I\cdot J + J\cdot I) = 0$. Therefore $N_\alpha^\beta := \{(x,x) \mid x \in J\}$ is an ideal of (the ring) α. As above we define

$$(a,b)\ \Theta_\alpha^\beta\ (c,d)\ :\Leftrightarrow\ (a-c,b-d) \in N_\alpha^\beta$$

and find $\Theta_\alpha^\beta \geq \Delta_\alpha^\beta$ implying $[I,J] = 0$, proving the claim.

6.3 Lattices: If α and β are congruences on a lattice $\underline{L}$ then $[\alpha,\beta] = \alpha\wedge\beta$.

(This is more generally true for every congruence distributive variety).

Namely consider $(x,y) \in \alpha\wedge\beta$ and look at Δ_α^β, π_1 and π_2, all thought of as congruences on α. Then

$$(x,x)\ \Delta_\alpha^\beta\ (y,y)\ \pi_2\ (x,y) \quad \text{and}$$

$$(x,x)\ \Delta_\alpha^\beta \vee \pi_1\ (x,y) \quad \text{hence}$$

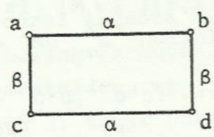
$$(x,x)\ \Delta_\alpha^\beta \vee (\pi_1\wedge\pi_2)\ (x,y)$$

by distributively and consequently $x\ [\alpha,\beta]\ y$ (since $\pi_1\wedge\pi_2 = 0$). On the other hand $[\alpha,\beta] \leq \alpha\wedge\beta$ follows from the definition of the commutator.

Thus we have seen that adding operations to our algebras increases the commutator until it is maximal when the algebras become congruence distributive. Congruence distributive varieties are in fact characterized amongst other modular varieties by this property as a corollary to 6.12 below.

Let us now return to develop the general theory of commutators.

6.4 Properties of Δ_α^β:



(i) $(a,b)\ \Delta_\alpha^\beta\ (c,d)$ implies

(ii) $(a,b)\ \Delta_\alpha^\beta\ (c,d)$ implies $(b,a)\ \Delta_\alpha^\beta\ (d,c)$

(iii) $a\ \beta\ b$ implies $(a,a)\ \Delta_\alpha^\beta\ (b,b)$.

Proof: (iii) being part of the definition, (ii) follows immediately from the symmetry of α, or fancier, note that $(x,y) \to (y,x)$ yields an automorphism of α, leaving invariant the generating set of Δ_α^β. For (i) note that $\Delta_\alpha^\beta \leq \beta\times\beta|_\alpha$ where $\beta\times\beta|_\alpha$ is the congruence on α given by $(x,y)\ \beta\times\beta|_\alpha\ (z,u)$ if $x\ \beta\ z$ and $y\ \beta\ u$.

6.5 <u>Properties of $[\alpha,\beta]$:</u>

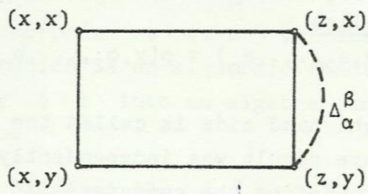
(i) $[\alpha,\beta] = \{(x,y) \mid (x,x)\ \Delta_\alpha^\beta\ (y,x)\}$

(ii) $[\alpha,\beta] = \{(x,y) \mid \exists z\ ((z,x)\ \Delta_\alpha^\beta\ (z,y))\}$
$= \{(x,y) \mid \exists z\ ((x,z)\ \Delta_\alpha^\beta\ (y,z))\}$
$= \{(x,y) \mid \exists z\ ((z,z)\ \Delta_\alpha^\beta\ (x,y))\}$.

(iii) $[\alpha,\beta]$ is a congruence relation on $\underline{A}$.

(iv) $[\alpha,\beta] \le \alpha \wedge \beta$.

<u>Proof:</u> (i) follows from 6.4 (ii).
(ii) follows with the Shifting Lemma applied to



The last equality follows from 6.4 (i) and (iii). (iv) is immediate from the definition and from 6.4 (i). For (iii): All properties of a congruence relation are immediate with 6.4. For transitivity we use 6.5, namely $x\ [\alpha,\beta]\ y\ [\alpha,\beta]\ z$ implies $(x,y)\ \Delta_\alpha^\beta\ (y,y)\ \Delta_\alpha^\beta\ (z,y)$ hence $x\ [\alpha,\beta]\ z$ with 6.5.

From MAL'CEV's description of congruences generated by a binary (symmetric) relation (0.1) we readily obtain, using 6.5:

6.6 <u>An alternative description of the commutator:</u>

$(x,y) \in [\alpha,\beta]$ <=> there exist unary algebraic functions $\tau_o,\dots,\tau_n$ on α and $(s_o,u_o),\dots,(s_n,u_n) \in \beta$ with

$\exists z$ with $\tau_o(s_o,s_o) = (z,z)$ [or: $\tau_o(s_o,s_o) = (x,x)$]
$\tau_i(u_i,u_i) = \tau_{i+1}(s_{i+1},s_{i+1}),\quad 0 \le i < n$
$\tau_n(u_n,u_n) = (x,y)$.

This description may of course be formulated coordinatewise. A unary algebraic function τ_i on α is nothing else than a pair $(t_i(x,\tilde{a}^i),t_i(x,\tilde{b}^i))$, where the t_i are term functions on $\underline{A}$ and $\tilde{a}^i = (a_1^i,\dots,a_n^i)$, $\tilde{b}^i = (b_1^i,\dots,b_n^i)$ are vectors componentwise congruent modulo α.
Note that the first line says no nore than

$$t_o(s_o,\tilde{a}^1) = t_o(s_o,\tilde{b}^1).$$

In particular it follows:

$$t_o(u_o,\tilde{a}^1)\ [\alpha,\beta]\ t_o(u_o,\tilde{b}^1).$$

This comes from 6.6 for $n = 0$. Since n may be arbitrary, this process can be iterated, defining $[\alpha,\beta]$ again as in 6.8 below. The case where we don't get started is

6.7 <u>A syntactical description of $[\alpha,\beta] = 0$:</u>

$[\alpha,\beta] = 0 \iff$ for all term functions $p(x_1,\dots,x_n)$ on $\underline{A}$ and $(a_2,b_2),\dots,(a_n,b_n) \in \alpha$ and $(x,y) \in \beta$ we have
$p(x,a_2,\dots,a_n) = p(x,b_2,\dots,b_n)$
<u>implies</u>
$p(y,a_2,\dots,a_n) = p(y,b_2,\dots,b_n)$.

The condition on the right hand side is called the <u>"term condition"</u> by several authors. The above result was independently found by R. McKENZIE who proposes to use it to define the commutator in nonmodular varieties (see 6.8 below). The first place where a similar condition is studied in the connection with congruences on direct products seems to be in WERNER [40], theorem 9. His assumptions nevertheless are far too strong and treat only the special case of simple algebras.

The condition also comes up in a totally different (so it seems) setting. We refer the reader to the interesting paper by FREESE, LAMPE and TAYLOR [12]. W. TAYLOR in [38] also showed that semigroups satisfying the above condition for $\alpha = \beta = 1$ are medial (which is only a minor part of [38] and not hard to show).

As we have indicated before, 6.7 actually can be turned into a definition of the commutator as

6.8 <u>Theorem:</u> $[\alpha,\beta]$ is the smallest congruence relation on $\underline{A}$ such that for all term functions t, $(x,y) \in \beta$ $(a_2,b_2),\dots,(a_n,b_n) \in \alpha$ we have

$t(x,a_2,\dots,a_n)\ [\alpha,\beta]\ t(x,b_2,\dots,b_n)$
<u>implies</u>
$t(y,a_2,\dots,a_n)\ [\alpha,\beta]\ t(y,b_2,\dots,b_n)$.

There is also a more geometric way to see 6.7. For this recall that by 0.1 a set $S \subseteq A$ is a congruence class for some congruence Θ on $\underline{A}$, iff for all unary algebraic functions τ of $\underline{A}$, we have: If one element of S is mapped back into S by τ then this is true for every element of S. Thus the right hand side of 6.7 states precisely that the sets $\delta_x^\beta = \{(y,y) \mid x\ \beta\ y\}$ are classes of some congruence on the algebra α. If so, they are certainly classes of Δ_α^β, hence $[\alpha,\beta] = 0$. On the other hand, if it was not a class of some congruence, we could assume $(x,x)\ \Delta_\alpha^\beta\ (u,v)$ for some $u \neq v$, yielding $(u,u)\ \Delta_\alpha^\beta\ (u,v)$ with 6.5 and hence $(u,v) \in [\alpha,\beta]$. This was our original proof of 6.7 in [19].

Notice that replacing equality signs in 6.7 by $\equiv$ (mod $[\alpha,\beta]$) is another way to 6.8. The left hand side then becomes a tautology, making the right hand side universally true. The justification for this will be given in 6.17 below.

Another conclusion is immediate from 6.6:

6.9 <u>Proposition:</u> Let $\phi: \underline{A} \to \underline{B}$ be a homomorphism and α,β congruences on $\underline{A}$. Then $\vec{\phi}[\alpha,\beta] \leq [\vec{\phi}\alpha,\vec{\phi}\beta]$.

<u>Proof:</u> Consider the homomorphism $\phi \times \phi: \alpha \to \vec{\phi}\alpha$ and apply it to the equations of 6.6. Each τ_i, which is an algebraic function of (the algebra) α will be transformed by $\phi \times \phi$ into an algebra function $\overline{\tau}_i$ of (the algebra) $\vec{\phi}\alpha$. Thus we get

$$\overline{\tau}_o(\phi(s_o),\phi(s_o)) = (\phi(x),\phi(x))$$
$$\overline{\tau}_i(\phi(u_i),\phi(u_i)) = \overline{\tau}_{i+1}(\phi(s_{i+1}),\phi(s_{i+1})), \quad \text{for } 0 \leq i < n$$
$$\overline{\tau}_n(\phi(u_n).\phi(u_n)) = (\phi(x),\phi(y)).$$

Hence with 6.6 we have $(\phi(x),\phi(y)) \in [\vec{\phi}\alpha,\vec{\phi}\beta]$.

Well known for groups, we get the following corollary:

6.10 <u>Corollary:</u> The commutator of fully invariant congruences is again fully invariant.

As we go on we need the following technical result:

6.11 <u>Theorem:</u> Let $\underline{D}$ be a subalgebra $\underline{A} \times \underline{A}$ (with $\mathrm{Con}(\underline{D})$ modular). Let κ_i, $i \in I$ be a family of congruence relations on $\underline{D}$ with the property $(x,y)\ \kappa_i\ (z,u) \Rightarrow (x,x)\ \kappa_i\ (z,z)$. Then for all $x,y,z \in \underline{D}$ we have:

$$(x,x) \bigvee_{i\in I} \kappa_i\ (y,z) \Rightarrow (y,y) \bigvee_{i\in I} (\kappa_i \wedge \pi_1)\ (y,z).$$

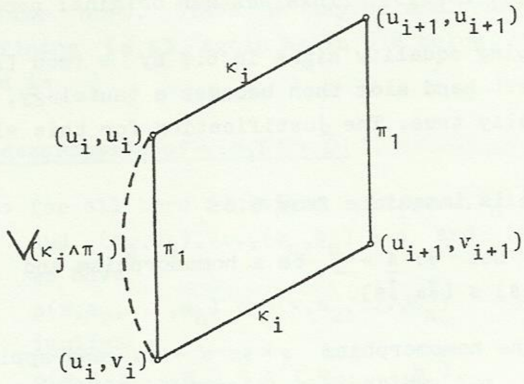
<u>Proof:</u> If $(x,x) \bigvee \kappa_j\ (y,z)$ then there exist w.l.o.g. $\kappa_o,\ldots,\kappa_{n-1}$ and $(u_o,v_o),\ldots,(u_n,v_n) \in \underline{D}$ with

$$(u_o,v_o) = (x,x), \quad (u_n,v_n) = (y,z) \quad \text{and}$$
$$(u_i,v_i)\ \kappa_i\ (u_{i+1},v_{i+1}) \quad \text{for } 0 \leq i < n.$$

By induction we show that

$$(u_i,u_i) \bigvee(\kappa_j \wedge \pi_1)\ (u_i,v_i).$$

Indeed this is trivial for $i = 0$. In passing from i to $i+1$ we note that $(u_i,u_i)\ \kappa_i\ (u_{i+1},u_{i+1})$ and, using the induction hypothesis we have the situation:



Thus the induction step is achieved with the Shifting Lemma. Setting now $i = n$ the theorem is proved.

As a corollary we have one of the most important properties of the commutator, namely join-distributivity. This was discovered by HAGEMANN and HERRMANN [24].

6.12 __Corollary:__ $[\alpha, \bigvee\beta_i] = \bigvee[\alpha,\beta_i]$.

__Proof:__ $\geq$ is clear since $\beta_i \leq \bigvee\beta_i$.

Trivially $\Delta_\alpha^{\bigvee\beta_i} = \bigvee\Delta_\alpha^{\beta_i}$. Hence supposing $(x,y) \in [\alpha,\bigvee\beta_i]$, i.e. $(x,x)\ \bigvee\Delta_\alpha^{\beta_i}\ (x,y)$ we conclude with 6.11 the relation $(x,x)\ \bigvee(\Delta_\alpha^{\beta_i} \wedge \pi_1)\ (x,y)$ which clearly means $(x,y) \in \bigvee[\alpha,\beta_i]$.

A second application of 6.11 yields a result of R. FREESE and R. McKENZIE [13]:

6.13 __Theorem:__ Let $\phi: \underline{A} \twoheadrightarrow \underline{B}$ be an onto homomorphism and α,β congruences on $\underline{B}$. Then $\overleftarrow{\phi}[\alpha,\beta] = [\overleftarrow{\phi}\alpha,\overleftarrow{\phi}\beta] \vee \ker\phi$.

__Proof:__ Using 6.9 we get that $(x,y) \in [\overleftarrow{\phi}\alpha,\overleftarrow{\phi}\beta]$ implies $(\phi(x),\phi(y)) \in [\phi\overleftarrow{\phi}\alpha,\phi\overleftarrow{\phi}\beta] = [\alpha,\beta]$ because ϕ is onto. For the reverse inclusion suppose $(a,b) \in \overleftarrow{\phi}[\alpha,\beta]$ i.e. $(x,y) \in [\alpha,\beta]$ with $x = \phi(a)$ and $y = \phi(b)$. The last relation can be written down as in 6.6. Since ϕ is onto there exist $(\overline{s}_i,\overline{t}_i) \in \overleftarrow{\phi}\beta$ with $(\overline{s}_i) = s_i$ and $(\overline{t}_i) = t_i$ and there are similarly algebraic functions $\overline{\tau}_i$ on $\overleftarrow{\phi}\alpha$ which arise from the given τ_i by replacing any constant (i.e. an element of α) by an arbitrary preimage under $\phi\times\phi$ (i.e. an element of $\overleftarrow{\phi}\alpha$). Since $\phi\times\phi$ is a homomorphism we obtain:

$\bar{\tau}_0(\bar{s}_0,\bar{s}_0)$ ker $\phi\times\phi$ (a,a)

$\bar{\tau}_i(\bar{t}_i,\bar{t}_i)$ ker $\phi\times\phi$ $\bar{\tau}_{i+1}(\bar{s}_{i+1},\bar{s}_{i+1})$ for $0 \le i < n$

$\bar{\tau}_n(\bar{t}_n,\bar{t}_n)$ ker $\phi\times\phi$ (a,b)

Hence $(a,a)\ \Delta^{\overleftarrow{\phi}\beta}_{\overleftarrow{\phi}\alpha} \vee$ ker $\phi\times\phi$ (a,b). Application of 2.6 with $\kappa_1 = \Delta^{\overleftarrow{\phi}\beta}_{\overleftarrow{\phi}\alpha}$ and $\kappa_2 =$ ker $\phi\times\phi$ yields

$$(a,a)\ (\Delta^{\overleftarrow{\phi}\beta}_{\overleftarrow{\phi}\alpha} \wedge \pi_1) \vee (\ker \phi\times\phi \wedge \pi_1)\ (a,b)$$

which immediately gives the missing inclusion.

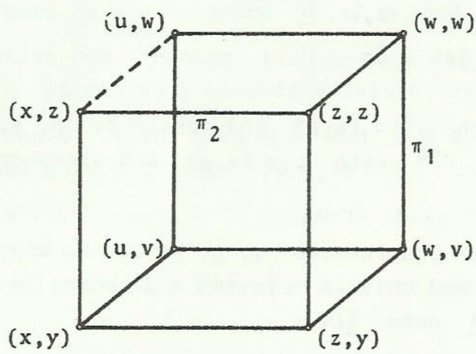
Another important property of the commutator is commutativity, see [24]. To prove it we use the Cube Lemma to imitate SMITH's proof of the permutable case.

6.14 <u>Theorem:</u> $[\alpha,\beta] = [\beta,\alpha]$.

<u>Proof:</u> Let us define $\bar{\Delta}^{\beta}_{\alpha} := \{((x,y),(u,v)) \mid (x,u)\ \Delta^{\beta}_{\alpha}\ (y,v)\}$. Clearly $(x,y) \in [\alpha,\beta]$ implies $(x,x)\ \bar{\Delta}^{\beta}_{\alpha}\ (x,y)$, hence we are done if we can show that $\bar{\Delta}^{\beta}_{\alpha} = \Delta^{\alpha}_{\beta}$.

Obviously $\bar{\Delta}^{\beta}_{\alpha}$ is a binary relation on β containing $((x,x),(y,y))$ whenever $(x,y) \in \alpha$. Reflexivity and symmetry are precisely properties 6.4 (iii) and 6.4 (ii).

For transitivity suppose $(x,u)\ \bar{\Delta}^{\beta}_{\alpha}\ (y,v)\ \bar{\Delta}^{\beta}_{\alpha}\ (z,w)$ which with the aid of 6.4 provides the following relations for Δ^{β}_{α}:



The Cube Lemma thus yields $(x,z)\ \Delta^{\beta}_{\alpha}\ (u,w)$ i.e. $(x,u)\ \bar{\Delta}^{\beta}_{\alpha}\ (z,w)$. Compatibility of $\bar{\Delta}^{\beta}_{\alpha}$ is trivially seen, hence $\bar{\Delta}^{\beta}_{\alpha}$ is a congruence relation on β, containing $((x,x),(y,y))$ whenever $x\ \alpha\ a$. Therefore $\bar{\Delta}^{\beta}_{\alpha} \ge \Delta^{\alpha}_{\beta}$. We conclude $\Delta^{\alpha}_{\beta} = \bar{\bar{\Delta}}^{\alpha}_{\beta} \ge \bar{\Delta}^{\beta}_{\alpha} \ge \Delta^{\alpha}_{\beta}$ which results in $\Delta^{\alpha}_{\beta} = \bar{\Delta}^{\beta}_{\alpha}$.

6.15 <u>Theorem ([24]):</u> Let α, β and γ be congruences on $\underline{B}$. Then $[\alpha,\beta] \le \gamma \le \alpha\wedge\beta$ if and only if there exists an algebra $\underline{A}$, a homomorphism $\phi\colon \underline{A} \twoheadrightarrow \underline{B}$ and congruences σ and τ on $\underline{A}$ such that

(1) $\sigma \wedge \tau \leq \overleftarrow{\phi}\gamma$

(2) $\sigma \vee \overleftarrow{\phi}\gamma \geq \overleftarrow{\phi}\alpha$ and

(3) $\tau \vee \overleftarrow{\phi}\gamma \geq \overleftarrow{\phi}\beta$.

<u>Proof:</u> "$\rightarrow$": $[\sigma \vee \overleftarrow{\phi}\gamma, \tau \vee \overleftarrow{\phi}\gamma] \leq [\sigma,\tau] \vee \overleftarrow{\phi}\gamma \leq \overleftarrow{\phi}\gamma$ applying 6.12, 6.14 and 6.5. Hence $[\overleftarrow{\phi}\alpha\ \overleftarrow{\phi}\beta] \leq \overleftarrow{\phi}\gamma$. From 6.13 we get $\overleftarrow{\phi}[\alpha,\beta] = [\overleftarrow{\phi}\alpha,\overleftarrow{\phi}\beta] \vee \ker \phi \leq \overleftarrow{\phi}\gamma$ and after applying $\overrightarrow{\phi}$ the result follows.

For the other direction take $\underline{A} := \alpha$ and $\phi := \pi_1: \underline{A} \rightarrow \underline{B}$ as given by $\pi_1(x,y) := x$. Define $\sigma := \ker \pi_2$ and $\tau := \Delta_\alpha^\beta$. Then $(x,y)\ \sigma \wedge \tau\ (u,v)$ implies $y = v$ and hence $x\ [\alpha,\beta]\ u$, hence $((x,y),(u,v)) \in \overleftarrow{\phi}[\alpha,\beta] \leq \overleftarrow{\phi}\gamma$. $(x,y)\ \overleftarrow{\phi}\alpha\ (u,v)$ implies $(x,u) \in \alpha$ hence $(x,v) \in \alpha$. Thus $(x,y)\ \overleftarrow{\phi}0\ (x,v)\ \sigma\ (u,v)$ hence $\overleftarrow{\phi}\alpha \leq \overleftarrow{\phi}0 \vee \sigma \leq \overleftarrow{\phi}\gamma \vee \sigma$. $(x,y)\ \overleftarrow{\phi}\beta\ (u,v)$ implies $x\ \beta\ u$ hence $(x,y)\ \overleftarrow{\phi}0\ (x,x)\ \Delta_\alpha^\beta\ (u,u)\ \overleftarrow{\phi}0\ (u,v)$ so $\overleftarrow{\phi}\beta \leq \overleftarrow{\phi}0 \vee \Delta_\alpha^\beta \leq \overleftarrow{\phi}\gamma \vee \tau$.

Inspection of the above proof leads to the following corollary which could also be used to define the commutator in nonmodular varieties. This corollary is due to HERMANN (unpublished).

6.16 <u>Corollary:</u> The commutator operation is the biggest binary operation $\langle , \rangle$ on the congruence lattices of algebras in a modular variety satisfying:

1. $\langle \alpha,\beta \rangle \leq \alpha \wedge \beta$
2. $\langle \alpha,\beta \vee \gamma \rangle = \langle \alpha,\beta \rangle \vee \langle \alpha,\gamma \rangle$
3. $\langle \alpha \vee \beta,\gamma \rangle = \langle \alpha,\gamma \rangle \vee \langle \beta,\gamma \rangle$
4. $\overleftarrow{\phi}\langle \alpha,\beta \rangle \leq \langle \overleftarrow{\phi}\alpha,\overleftarrow{\phi}\beta \rangle \vee \overleftarrow{\phi}0$.

<u>Proof:</u> Use 6.15 with $\gamma := [\alpha,\beta]$ and construct $\underline{A}$, ϕ, σ and τ. Then repeat the proof of "$\leftarrow$" with $\gamma = [\alpha,\beta]$ and elsewhere $[,]$ replaced by $\langle , \rangle$.

6.17 <u>Corollary:</u> For congruences α, β, γ of the algebra $\underline{A}$ we have that $[\alpha,\beta] \leq \gamma$ if and only if $[\overrightarrow{\phi}\alpha,\overrightarrow{\phi}\beta] = 0$ where ϕ is the canonical homomorphism from $\underline{A}$ onto $\underline{A}/\gamma$.

<u>Proof:</u> $[\overrightarrow{\phi}\alpha,\overrightarrow{\phi}\beta] = 0$ implies with 6.9 that $\overrightarrow{\phi}[\alpha,\beta] = 0$ which is equivalent to $[\alpha,\beta] \leq \ker \phi \leq \gamma$. On the other hand $\overleftarrow{\phi}[\overrightarrow{\phi}\alpha,\overrightarrow{\phi}\beta] = [\overleftrightarrow{\phi\phi}\alpha,\overleftrightarrow{\phi\phi}\beta] \vee \ker \phi = [\alpha \vee \ker \phi,\beta \vee \ker \phi] \vee \ker \phi \leq$ $\leq [\alpha,\beta] \vee \ker \phi$. Assuming $[\alpha,\beta] \leq \gamma = \ker \phi$ we get that $[\overrightarrow{\phi}\alpha,\overrightarrow{\phi}\beta] = 0$.

SMITH [36] says that "α centralizes β" if $[\alpha,\beta] = 0$. 6.17 shows that it is enough to know the relation of centralizing to describe the commutator operation. Moreover the results obtained so far allow the notion of the "centralizer" of a congruence α, which is defined to be the largest congruence $\zeta(\alpha)$ which centralizes α. The existence is guaranteed by

6.12 and

$$\zeta(\alpha) = \{\beta \mid [\alpha,\beta] = 0\}.$$

$\zeta := \zeta(1)$ is called the center of $\underline{A}$.
Recently the importance of the centralizer concept became clear through a beautiful theorem by HRUSHOVSKII, which generalized the important Jonsson Lemma [28] and its subsequent generalizations due to HAGEMANN, HERRMANN [24] and FREESE, McKENZIE [13].

6.18 Theorem (HRUSHOVSKII): Let $\underline{S} \in HSP(\mathcal{K})$ be subdirectly irreducible with monolith μ. Then $\underline{S}/_{\zeta(\mu)} \in HSP_u(\mathcal{K})$.

Here $P_u(\mathcal{K})$ is the class of all ultraproducts of members of $\mathcal{K}$. Note that in congruence-distributive varieties $\zeta(\mu) = 0$ so $\underline{S} \in HSP_u(\mathcal{K})$ as stated in Jonsson's Lemma. See [1] for the definition and the important properties of ultraproducts. We prove the theorem here in a slightly more general form. Recall that an algebra $\underline{S}$ is called finitely subdirectly irreducible if the smallest congruence $0_{\underline{S}}$ is not an intersection of finitely many congruences above $0_{\underline{S}}$.

6.19 Theorem: Let $\underline{S} \in HSP(\mathcal{K})$ be finitely subdirectly irreducible and define $\xi := \bigvee\{\zeta(\alpha) \mid \alpha \neq 0\}$. Then $\underline{S}/_{\xi} \in HSP_u(\mathcal{K})$.

Note that in 6.18 ξ coincides with $\zeta(\mu)$, so 6.19 implies 6.18. The proof is very similar to Jonsson's proof in [28] and it is based on W. LAMPE's proof of HRUSHOVSKII's theorem.
Since $\underline{S} \in HSP(\mathcal{K})$ there is a subalgebra $\underline{U}$ of a product $\prod_{i \in I} \underline{A}_i$ with $\underline{A}_i \in \mathcal{K}$ and a surjective homomorphism $\phi: \underline{U} \twoheadrightarrow \underline{S}$. Let Θ be the kernel of ϕ in $Con(\underline{U})$. Θ is finitely meet-irreducible. First we show:

(§) Suppose $\alpha, \beta \in Con(U)$ with $\alpha \wedge \beta \leq \Theta$ and $\alpha \nleq \Theta$ and $\beta \nleq \Theta$. Then there exists a $\gamma \nleq \Theta$ with $[\alpha\vee\beta, \gamma] \leq \Theta$.

Take $\gamma := (\alpha\vee\Theta) \wedge (\beta\vee\Theta)$, then γ is properly above Θ since Θ is finitely meet-irreducible. Now
$[\alpha,\gamma] \leq [\alpha,\beta\vee\Theta] = [\alpha,\beta] \vee [\alpha,\Theta] \leq (\alpha\wedge\beta) \vee \Theta \leq \Theta$ and similarly $[\beta,\gamma] \leq \Theta$ so $[\alpha\vee\beta,\gamma] \leq \Theta$.

For subsets D of I define a congruence relation η_D on $\underline{U}$ by
$x \, \eta_D \, y$ iff $\{i \mid x(i) = y(i)\} \supseteq D$.
Let $\mathcal{F}$ be a filter on I, maximal with respect to the condition that $\eta_D \leq \Theta$ for all $D \in \mathcal{F}$. Let $\mathcal{U}$ be any ultrafilter extending $\mathcal{F}$. We claim:

$$\forall_{E \in \mathcal{U}} \quad \eta_E \leq \overleftarrow{\phi}(\xi).$$

If $E \in \mathcal{F}$ then this is clear since $\eta_E \leq \Theta \leq \overleftarrow{\phi}(\xi)$. If $E \notin \mathcal{F}$ then by the maximality of $\mathcal{F}$ there exists a $G \in \mathcal{F}$ with $\eta_{E\cap G} \nleq \Theta$ and

$\eta_{E'\cap G} \not\leq \theta$. But obviously $\eta_{E\cap G} \wedge \eta_{E'\cap G} = \eta_G \leq \theta$. Now with (§) we find a γ properly above θ with $\eta_E \leq \eta_{E\cap G} \vee \eta_{E'\cap G} \leq \zeta(\gamma:\theta)$ (where $\zeta(\gamma:\theta)$ denotes the greatest congruence whose commutator with γ is below θ). Since finally $\zeta(\gamma:\theta) \leq \overset{+}{\phi}(\xi)$ we have that $\eta_E \leq \overset{+}{\phi}(\xi)$. So the ultrafilter congruence η given by $\bigvee\{\eta_E \mid E \in \mathcal{U}\}$ is below $\overset{+}{\phi}(\xi)$. With standard arguments (see [28]) now $\underline{S}/_\xi \in HSP_u(\mathcal{K})$.

Let us introduce prime and semiprime congruences.

6.20 Definition: A congruence relation θ on the algebra $\underline{A}$ is called prime if for any two congruences α, β on $\underline{A}$ we have: $[\alpha,\beta] \leq \theta$ implies $\alpha \leq \theta$ or $\beta \leq \theta$. θ is called semiprime, if the above implication holds in the special cases where $\alpha = \beta$.

Clearly prime congruences are finitely meet irreducible moreover they are precisely the fixed points of the following mapping ξ: $Con(\underline{A}) \to Con(\underline{A})$:

$$\theta \to \xi(\theta) := \bigvee\{\alpha \mid \exists_{\beta \not\leq \theta} \; [\alpha,\beta] \leq \theta\}.$$

Note that for (nonassociative) rings $\underline{R}$ and ideals I,J,K the above notion of primeness is equivalent to the usual notion using products of ideals, i.e. K is prime iff for all I,J ideals of R we have: $I \cdot J \leq K$ implies $I \leq K$ or $J \leq K$. This is not hard to show.

The intersection of prime congruences is semiprime. The converse is due to KEIMEL [44]: Every semiprime congruence is the intersection of prime congruences. It is useful to introduce the notation $\sqrt{\theta}$ for the intersection of all prime congruences above θ. Then KEIMEL's theorem says that θ is semiprime iff $\sqrt{\theta} = \theta$. We write $\sqrt{\underline{A}}$ for $\sqrt{0_{\underline{A}}}$ and call $\sqrt{\underline{A}}$ the prime radical of $\underline{A}$. From 6.19 we obtain

6.21 Theorem: Let $\underline{A}$ be an algebra in a modular variety $HSP(\mathcal{K})$. Then $\underline{A}/\sqrt{\underline{A}} \in P_s HSP_u(K)$.

7. TERNARY TERMS FOR MODULARITY

In the last chapter we have not fully made use of the fact that we are in a modular variety. We could have done with the hypothesis that congruences on $\underline{A}$, when considered as subalgebras of $\underline{A} \times \underline{A}$ have modular congruence lattices.

In this chapter we will use the ternary term $t(x,y,z)$ as constructed in theorem 4.3 to obtain an important permutability formula for congruences. This in turn will be the key for the construction of ternary terms which may replace the quaternary DAY-terms.

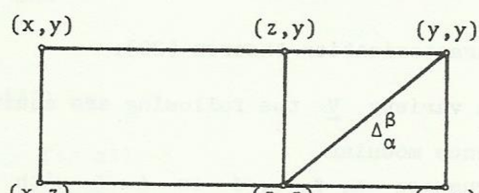
The discovery of those terms was surprising for various reasons. Firstly, since ternary terms are describing properties of three generated algebras, the existence of ternary terms seems to contradict the fact that there exist varieties, all of whose three-generated algebras are modular, but the variety as a whole is not congruence modular. The solution however is, that our terms are in fact describing a property which is strictly stronger than modularity and which is shared automatically by every algebra which is contained in a modular variety.

Secondly, the terms we produce are just B. JONSSON's terms for distributivity (thm. 1.4) and A.I. MAL'CEV's term for permutability (thm. 1.2) "glued" together. Thus giving account of the fact that modular varieties are somewhere in between (and including) permutable varieties on the one hand and distributive varieties on the other hand.

This general principle will come up time and again in the later chapters. Investigation of the commutators will usually indicate in which direction to go.

Let us now start with investigating the rôle of the ternary term $t(x,y,z)$ from 4.3 within commutator theory.

To this end let α and β be congruences on $\underline{A}$ with $\alpha \geq \beta$. Suppose x,y,z are elements from $\underline{A}$ with $x \; \alpha \; y \; \beta \; z$. We set $\Psi := \Delta_\alpha^\beta$ and apply the term t to the situation (inside the algebra α):



Using 4.3 we find

(§) $(x,y)\ \Delta_\alpha^\beta\ (t(x,y,z),z)$.

In particular, setting $x=y$, we obtain:

$$t(x,x,z)\ [\alpha,\beta]\ z.$$

The interesting case is where $\alpha=\beta$:

7.1 <u>Lemma:</u> There is a term t in every modular variety $\underline{V}$ such that

$t(x,y,y)=x$ is an equation in $\underline{V}$

and $t(a,a,b)\ [\alpha,\alpha]\ b$ holds for $a\ \alpha\ b$.

An important application, which has also been found independently by W. TAYLOR is:

7.2 <u>Lemma:</u> For any congruences α and β the formulas

$\Theta\circ\Psi \subseteq [\Theta,\Theta]\circ\Psi\circ\Theta$

and $\Theta\circ\Psi \subseteq \Psi\circ\Theta\circ[\Psi,\Psi]$

are true.

<u>Proof:</u> For $(x,z)\in\Theta\circ\Psi$ there exists a y with $x\ \Theta\ y\ \Psi\ z$. Then
$x\ [\Theta,\Theta]\ t(y,y,x)\ \Psi\ t(z,y,x)\ \Theta\ t(z,y,y)=z$ and
$x=t(x,y,y)\ \Psi\ t(x,y,z)\ \Theta\ t(y,y,z)\ [\Psi,\Psi]\ z$ by 7.1.

Our plan is now, to use 7.2 for obtaining a new Mal'cev-type condition for congruence modularity. To this end, the commutator has to be removed from the formulas in 7.2, since we have not defined it for nonmodular varieties. We find the desired version of 7.2 in the following lemma:

7.3 <u>Lemma:</u> For congruences α, β, γ and δ with $\alpha\le\gamma\vee\delta$ we have $\alpha\circ\beta\subseteq((\alpha\wedge\gamma)\vee(\alpha\wedge\delta))\circ\beta\circ\alpha$.

<u>Proof:</u>
$$\begin{aligned}\alpha\circ\beta &\subseteq [\alpha,\alpha]\circ\beta\circ\alpha \subseteq\\ &\subseteq [\alpha,\gamma\vee\delta]\circ\beta\circ\alpha\subseteq\\ &\subseteq ([\alpha,\gamma]\vee[\alpha,\delta])\circ\beta\circ\alpha\subseteq\\ &\subseteq ((\alpha\wedge\gamma)\vee(\alpha\wedge\delta))\circ\beta\circ\alpha.\end{aligned}$$

Finally then the characterization theorem [20]:

7.4 <u>Theorem:</u> For a variety $\underline{V}$ the following are equivalent:

(i) $\underline{V}$ is congruence modular.

(ii) For all congruences α, β, γ, δ on $\underline{A}\in\underline{V}$ with $\gamma\vee\delta\ge\alpha$ the formula $\alpha\circ\beta\subseteq((\alpha\wedge\gamma)\vee(\alpha\wedge\delta))\circ\beta\circ\alpha$ holds.

(iii) For all congruences α, β, γ on $\underline{A}\in\underline{V}$ with $\gamma\vee\beta\ge\alpha$ the formula $\alpha\circ\beta\subseteq((\alpha\wedge\beta)\vee(\alpha\wedge\gamma))\circ\beta\circ\alpha$ holds.

(iv) For some $n \in \mathbb{N}$ there exist ternary terms $q_o, \ldots, q_n$ and p such that the following equations are true in $\underline{V}$:

(1) $q_o(x,y,z) = x$

(2) $q_i(x,y,x) = x$ for all $0 \leq i < n$

(3) $q_i(x,x,y) = q_{i+1}(x,x,y)$ for i even

(4) $q_i(x,y,y) = q_{i+1}(x,y,y)$ for i odd

(5) $q_n(x,y,y) =$

$= p(x,y,y)$

(6) $p(x,x,y) = y$

(1)–(5): JONSSON-terms; (5)–(6): MAL'CEV-term

Comment: It may be interesting to notice that p could be trivial (i.e. a projection). In this case p would be the third projection and $q_n(x,y,y)$ would be equal to y. We may suppose that n is odd, for otherwise we have $q_{n-1}(x,y,y) = y$ as well. Now define $q_{n+1}(x,y,z) = z$. Now the equations we are left with are precisely B. JONSSON's equations showing that $\underline{V}$ is congruence distributive (1.4).

On the other hand, if all the q_i's are projections, this would imply $q_i(x,y,z) = x$ for every i. Hence what we are left with are the equations which are precisely those of MAL'CEV witnessing permutability (1.2).

Proof of 7.4: (i) → (ii) → (iii) is lemma 7.3.
For (iii) → (iv) let $\underline{F}_{\underline{V}}(3)$ be the free algebra in $\underline{V}$ generated by $X = \{x,y,z\}$. Consider the congruences $\Theta_{(x,y)}$, $\Theta_{(y,z)}$ and $\Theta_{(x,z)}$ which are generated by the nontrivial partitions of X.

Clearly $\Theta_{(y,z)} \vee \Theta_{(x,z)} \geq \Theta_{(x,y)}$. Hence (iii) tells us:

$$\Theta_{(x,y)} \circ \Theta_{(y,z)} \leq ((\Theta_{(x,y)} \wedge \Theta_{(x,z)}) \vee (\Theta_{(x,y)} \wedge \Theta_{(y,z)})) \circ \Theta_{(y,z)} \circ \Theta_{(x,y)}.$$

(x,z) is therefore in the right hand side, which implies that there exist elements $t_o, \ldots, t_n$, and r in $\underline{F}_{\underline{V}}(3)$ such that

(0) $x = t_o$

(2') $t_i \; \Theta_{(x,y)} \wedge \Theta_{(x,z)} \; t_{i+1}$ for i even

(3') $t_i \; \Theta_{(x,y)} \wedge \Theta_{(y,z)} \; t_{i+1}$ for i odd

(4) $t_n \; \Theta_{(y,z)} \; r$ and

(5) $r \; \Theta_{(x,y)} \; z$.

We rewrite (2') and (3') by

(1) $t_i \; \Theta_{(x,y)} \; x$ for all i

(2) $t_i \; \Theta_{(x,z)} \; t_{i+1}$ for i even

(3) $t_i \; \Theta_{(y,z)} \; t_{i+1}$ for i odd.

By the usual arguments then the t_i and r do correspond to ternary terms $\overline{q}_i$ and p such that (in accordance with (0), ..., (5)) the following equations are satisfied in $\underline{V}$:

(0) $x = \overline{q}_o(x,y,z)$

(1) $\overline{q}_i(x,x,y) = x$ for all $0 \leq i < n$

(2) $\overline{q}_i(x,y,x) = \overline{q}_{i+1}(x,y,x)$ for i even

(3) $\overline{q}_i(x,y,y) = \overline{q}_{i+1}(x,y,y)$ for i odd

(4) $\overline{q}_n(x,y,y) = p(x,y,y)$

(5) $p(x,x,y) = y$.

By simply redefining $q_i(x,y,z) := \overline{q}_i(x,z,y)$ we get the desired terms.

For (iv) → (i) it is enough to prove the Shifting Lemma according to 3.6. So we start with congruences α, β, γ with $\alpha \wedge \beta \leq \gamma$ and elements x, y, z, u such that $x \;\alpha\; z \;(\beta \wedge \gamma)\; u \;\alpha\; y \;\beta\; x$. Again we might as well assume that $\alpha \wedge \beta = 0$, otherwise we would have to replace equality signs by $\equiv$ (mod $\alpha \wedge \beta$).

Consider the following points of $\underline{A}$:

$\overline{p} := p(z,u,y)$

$\hat{q}_i := q_i(x,u,y)$

$\overline{q}_i := q_i(x,z,y)$ and

$\tilde{q}_n := q_n(z,y,u)$.

We obtain the following relations:

I $x \;\beta\; \overline{q}_i$ for all i

II $x \;\beta\; \hat{q}_i$ for all i, by using equation 2.

Equation 6 yields:

III $y \;(\beta \wedge \gamma)\; \overline{p}$.

Thus the $\overline{q}_i$, $\hat{q}_i$ and $\overline{p}$ lie on the β-line connecting x and y, in particular they are mutually β-congruent. Equations 3 and 4 now provide for

IV $\overline{q}_i \;\alpha\; \overline{q}_{i+1}$ for i even and

V $\hat{q}_i \;\alpha\; \hat{q}_{i+1}$ for i odd.

Hence with I and II we have

VI $\overline{q}_i = \overline{q}_{i+1}$ for i even and

VII $\hat{q}_i = \hat{q}_{i+1}$ for i odd.

But notice that by definition we have also

VIII $\overline{q}_i \;\gamma\; \hat{q}_i$ for every i.

This, together with VI and VII gives us:

$$x = \bar{q}_o = \bar{q}_1 \;\gamma\; \hat{q}_1 = \hat{q}_2 \;\gamma\; \bar{q}_2 = \bar{q}_3 \;\gamma\; \hat{q}_3 \ldots \hat{q}_n, \quad \text{i.e.}$$

IX $\quad x \;\gamma\; \hat{q}_n$. (No matter whether n is odd or even!)

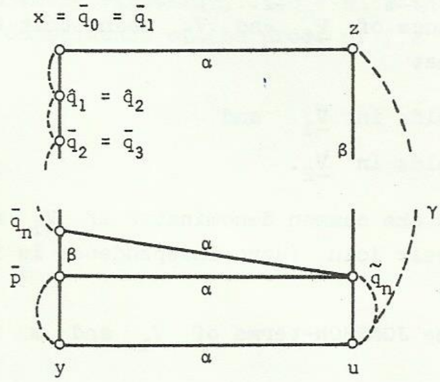
Hence in view of III, all we have to do is to show that $\hat{q}_n = \bar{p}$. For this reason notice that

$$\hat{q}_n = q_n(x,u,y) \;\alpha\; q_n(z,y,u) = \tilde{q}_n \quad \text{and}$$

$$\bar{p} = p(z,u,y) \;\alpha\; p(z,y,y) = q_n(z,y,y) \;\alpha\; q_n(z,y,u) = \tilde{q}_n.$$

Hence $\hat{q}_n \;\alpha\; \bar{p}$ and $\hat{q}_n \;\beta\; \bar{p}$ by I and III. Thus $\hat{q}_n = \bar{p}$ and we are finished.

To see what is really going on, a look at the following picture is worthwile. (Notice, that $\tilde{q}_n \;\gamma \wedge \beta\; u$ by equation 2.)



Of course, as a consequence of 7.4 one should be able to construct our terms $q_o,\ldots,q_n$ and p from the DAY-terms $m_o,\ldots,m_k$. In fact a constructive proof could be worked out following the ribbon backwards through the theory of commutators and through the construction of the sixary term from chapter 4 to the DAY-terms. This would be an enormous and tedious task, so messy that it is inconceivable that a proof of 7.4 would ever have been found without the theory of commutators at disposal.

For the simplest nontrivial case, i.e. if $\underline{V}$ is 3-permutable we have worked out the terms:

7.5 <u>Proposition:</u> Suppose $\underline{V}$ is a 3-permutable variety, where $r(x,y,z)$ and $s(x,y,z)$ are the terms for 3-permutability (1.3) i.e. satisfying

$$x = r(x,y,y)$$
$$r(x,x,y) = s(x,y,y)$$
$$s(x,x,y) = y.$$

Then the following are terms satisfying the equations of 7.4:

$$q_1(x,y,z) = s(r(s(x,z,z),y,x),r(x,y,z),x)$$
$$q_2(x,y,z) = r(r(s(x,z,z),z,x),r(s(x,z,z),y,x),r(x,y,z))$$
$$q_3(x,y,z) = r(s(x,z,z),z,x)$$
$$p(x,y,z) = r(s(x,y,z),y,x).$$

We can do better, namely have 7.4 hold with $n = 2$ because of 3-permutability. But the terms will get deeper nested as

$$\overline{q}_1(x,y,z) = r(x,q_1(x,y,z),q_2(x,y,z))$$
$$\overline{q}_2(x,y,z) = s(q_1(x,y,z),q_2(x,y,z),q_3(x,y,z))$$
$$p(x,y,z) = r(s(x,y,z),y,x).$$

Is there any shorter way?

Very beautifully the terms of 7.4 come up when we form the join of two independent varieties $\underline{V}_1$ and $\underline{V}_2$ where $\underline{V}_1$ is distributive and $\underline{V}_2$ is permutable. Independence of $\underline{V}_1$ and $\underline{V}_2$ means that there is a binary term $f(x,y)$ such that

$f(x,y) = x$ holds in $\underline{V}_1$ and

$f(x,y) = y$ holds in $\underline{V}_2$.

Clearly modularity is the common denominator of $\underline{V}_1$ and $\underline{V}_2$ and should hence be shared by their join. (Here independence is indispensable as a consequence of [15].)

Let $t_o,\dots,t_n$ be the JONSSON-terms of $\underline{V}_1$ and m the MAL'CEV-term of $\underline{V}_2$ then

$$q_i(x,y,z) := f(t_i(x,y,z),x) \quad \text{and}$$
$$p(x,y,z) := f(z,m(x,y,z))$$

are the terms for theorem 7.4.

There is, also an interesting converse to this construction. Namely, given the terms q_i and p of Theorem 7.4, define subvarieties of the given variety $\underline{V}$ by

$$\underline{V}_d := \mathrm{Mod}\,\{p(x,y,y) = y\} \cap \underline{V}$$
$$\underline{V}_p := \mathrm{Mod}\,\{p(x,y,y) = x\} \cap \underline{V}.$$

Then we obtain:

7.6 <u>Proposition:</u> $\underline{V}_d$ is congruence distributive, $\underline{V}_p$ is permutable and $\underline{V}_d$ and $\underline{V}_p$ are independent subvarieties of $\underline{V}$.

If $\underline{V}$ as an example is the variety of generalized right complemented semigroups then $r(x,y,z) := x\cdot(y*z)$ and $s(x,y,z) := z\cdot(y*x)$ show that $\underline{V}$ is three-permutable. Thus the term $p(x,y,z)$ from theorem 7.4 is

$$p(x,y,z) = r(s(x,y,z),y,x) = (z\cdot(y*x))\cdot(y*x).$$

Thus $\underline{V}_d = \text{Mod}\ \{(y\cdot(y*x))\cdot(y*x) = y\} \cap \underline{V}$ and

$\underline{V}_p = \text{Mod}\ \{(y\cdot(y*x))\cdot(y*x) = x\} \cap \underline{V}$.

Let us add a remark about the connections between the different terms we have been using so far. It will become obvious in the following chapter (see thm. 8.5 below) that, on defining

$$t(x,y,z) := p(z,y,x)$$

we obtain a term satisfying all properties from our originally constructed ternary term from chapter 4. On the other hand, for every term $t(x,y,z)$ with the properties described in theorem 4.3,

$$p(x,y,z) := t(z,y,x)$$

yields a term for which there exist $q_0,\ldots,q_n$ with the properties of theorem 7.4. This is a consequence of the proof of 7.2.

8. PERMUTABILITY RESULTS

In this chapter we give criteria for congruences Θ and Ψ to permute. Some of the results improve corresponding findings from chapter 4. The formula on which most of this is based is the one from 7.2.

We start with

8.1 Theorem: Let $\underline{A}$ be an algebra in a modular variety and Θ and Ψ congruences on $\underline{A}$. Then the following statements are equivalent:

(i) Θ permutes with Ψ

(ii) $\Theta^{(n)}$ permutes with $\Psi^{(m)}$ for all $n,m \in \mathbb{N}$

(iii) $\Theta^{(n)}$ permutes with $\Psi^{(m)}$ for some $n,m \in \mathbb{N}$.

Here we use the following definition:

$$\Theta^{(o)} := \Theta, \quad \Theta^{(n+1)} := [\Theta^{(n)},\Theta^{(n)}].$$

Θ is called solvable, if $\Theta^{(n)} = 0$ for some natural number n.

Note that as a corollary to 8.1 we have a result from [21]:

8.2 Corollary: A solvable congruence relation permutes with every congruence relation.

Proof of 8.1: Iterating the formula $\Theta \circ \Psi \subseteq [\Theta,\Theta] \circ \Psi \circ \Theta$ and its symmetric form we obtain:

$$\Theta \circ \Psi \subseteq \Psi \circ \Theta^{(n)} \circ \Psi^{(m)} \circ \Theta \quad \text{for any } n,m \in \mathbb{N}.$$

Namely, by symmetry we are done if we show the induction step from n to $n+1$. Assuming the above formula we obtain

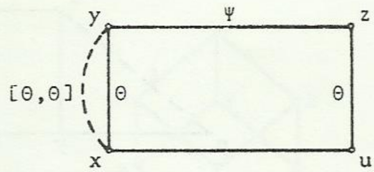
$$\begin{aligned} \Theta \circ \Psi &\subseteq \Psi \circ \Theta^{(n)} \circ \Psi^{(m)} \circ \Theta \subseteq \\ &\subseteq \Psi \circ [\Theta^{(n)},\Theta^{(n)}] \circ \Psi^{(m)} \circ \Theta^{(n)} \circ \Theta \subseteq \\ &\subseteq \Psi \circ \Theta^{(n+1)} \circ \Psi^{(m)} \circ \Theta. \end{aligned}$$

Now the above formula gives us (iii) $\to$ (i) since $\Theta^{(n)} \leq \Theta$ and $\Psi^{(m)} \leq \Psi$. It remains to prove (i) $\to$ (ii).

Again by induction we may assume we have already proven that $\Theta^{(n-1)}$ permutes with $\Psi^{(m)}$ hence have to show $\Theta^{(n)}$ permutes with $\Psi^{(m)}$. Changing notation, we have to show that $[\Theta,\Theta]$ permutes with Ψ in case Θ permutes with Ψ. Suppose $(x,z) \in [\Theta,\Theta] \circ \Psi$, i.e. for some y we have

$$x \; [\Theta,\Theta] \; y \; \Psi \; z.$$

In particular $(x,y) \in \Theta$ hence $x\ \Theta\ y\ \Psi\ z$. Since Θ and Ψ permute we find a u with

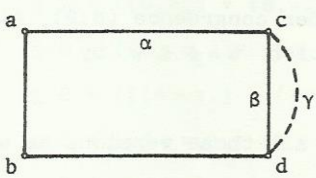


The Shifting Lemma gives us $(u,z) \in (\Theta \wedge \Psi) \vee [\Theta,\Theta]$. But since $[\Theta \wedge \Psi, \Theta \wedge \Psi] \leq [\Theta,\Theta]$, $(\Theta \wedge \Psi)^{(1)}$ permutes with $[\Theta,\Theta]$ which implies that $\Theta \wedge \Psi$ permutes with $[\Theta,\Theta]$ by the direction (iii) → (i).

Hence there exists an element w with $u\ \Theta \wedge \Psi\ w\ [\Theta,\Theta]\ z$, thus $x\ \Psi\ w\ [\Theta,\Theta]\ z$ which was to be shown.

As a corollary to the proof we get a stronger kind of Shifting Lemma, namely:

8.3 <u>Corollary:</u> Let α, β, γ be congruences on an algebra $\underline{A}$ in a modular variety such that $(\alpha \wedge \beta)^{(n)}$ permutes with $\gamma^{(m)}$ for some $n,m \in \mathbb{N}$. Then



implies $(a,b) \in (\alpha \wedge \beta) \circ \gamma$.

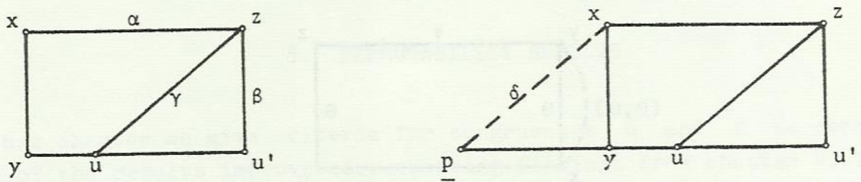
Iterating the formula of 7.2 in a way similarly as in the proof of 8.1 we find that joins of congruences may be easily computed, once the join of some of there iterated commutators are known:

8.4 <u>Lemma:</u> If Θ and Ψ are congruences and $n,m \in \mathbb{N}$, then

$$\begin{aligned}\Theta \vee \Psi &= \Theta \circ (\Theta^{(n)} \vee \Psi^{(m)}) \circ \Psi = \\ &= (\Theta^{(n)} \vee \Psi^{(m)}) \circ \Theta \circ \Psi = \\ &= \Theta \circ \Psi \circ (\Theta^{(n)} \vee \Psi^{(m)}).\end{aligned}$$

An application will be given in 8.8 below.
Theorem 4.3 may also be improved as follows:

8.5 <u>Theorem:</u> In every modular variety $\underline{V}$ there exists a ternary term $p(x,y,z)$ such that $p(x,x,y) = y$ is an equation of $\underline{V}$ and for congruences α, β and γ with $\alpha \wedge \beta$ permuting with γ we obtain



with $\delta = \gamma \circ (\alpha \wedge \beta)$ and $\underline{p} = p(u,u',y)$.

<u>Proof:</u> Define $\hat{q}_i$ and $\underline{p}$ precisely as in the proof of 7.4. Hence $x \ \gamma \vee (\alpha \wedge \beta) \ \hat{q}_n$ and finally $p(u,u',y) \ \gamma \ p(z,u',y)$ $p(u,u',y) \ \gamma \ p(z,u',y) \ (\alpha \wedge \beta) \ \hat{q}_n$ with $p(u,u',y) \ \alpha \ y$ yielding $x \ \gamma \circ (\alpha \wedge \beta) \ \underline{p}$.

With the help of the above finally 4.4 can be strengthened:

8.6 <u>Corollary:</u> Let α, β and γ be congruences such that $\gamma \leq \alpha \circ \beta = \beta \circ \alpha$ and $\gamma^{(n)}$ permutes with $(\alpha \wedge \beta)^{(m)}$ for some $n,m \in \mathbb{N}$. Then γ permutes with α (and with β).

<u>Remark:</u> The special case where $\alpha \wedge \beta \leq \gamma$ was the first result in this direction. It was proven in [17]. Combining this with the result that solvable congruences permute with every other congruence (8.2), A. WOLF gave a short argument to replace the condition $\alpha \wedge \beta \leq \gamma$ by $(\alpha \wedge \beta)^{(n)} \leq \gamma$.

Corollary 8.5 is its present form subsumes all those versions as well as 8.2, just set $\beta = 1$ and $m = 0$.

If α and β are congruences then modularity of the lattice of congruence relations implies that the maps $x \longrightarrow \alpha \vee x$ and $x \longrightarrow \beta \wedge x$ are isomorphisms between the intervals $\alpha \wedge \beta \leq x \leq \beta$ and $\alpha \leq x \leq \alpha \vee \beta$.

In [21] it was proved that

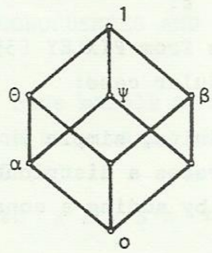
8.7 <u>Lemma:</u> If α and β permute, then the above isomorphisms preserve permutability of congruences.

<u>Proof:</u> For $\alpha \wedge \beta \leq \Theta, \Psi \leq \beta$ we find by 4.4 that Θ and Ψ permute with α. Hence $\alpha \vee \Theta$ $(= \alpha \circ \Theta)$ permutes with $\alpha \vee \Psi$ $(= \alpha \circ \Psi)$.

If $\alpha \leq \Theta, \Psi \leq \alpha \vee \beta$ and Θ permutes with Ψ then notice that by repeated use of 4.4 $\Theta \wedge \Psi$ permutes with $\beta \wedge (\Theta \vee \Psi)$. Thus we might as well assume that $\alpha \wedge \beta = 0$ and $\Theta \wedge \Psi = \alpha$. Moreover, instead of looking at each class of $\Theta \vee \Psi$ separately, we may assume $\Theta \vee \Psi = 1$.

Thus Θ and Ψ give a factor decomposition of $\underline{A}/\alpha$. Since α permutes with β, we have a direct decomposition of $\underline{A}$.

Therefore, all the congruences considered so far were factor-congruences, in particular, $\Theta \wedge \beta$ and $\Psi \wedge \beta$ permute.



Combining the preceding lemma with 8.4 we obtain a short proof of a theorem which is due to A. WOLF [42]:

8.8 <u>Theorem:</u> If $\underline{A}$ and $\underline{B}$ are algebras in a modular variety such that $\underline{A}$ and $\underline{B}$ have permutable congruences, then so has $\underline{A} \times \underline{B}$.

<u>Proof:</u> For Θ, Ψ congruences on $\underline{A} \times \underline{B}$ we have by 8.4:

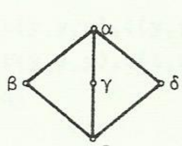
$$\begin{aligned}
\underline{\Theta \vee \Psi} &= \Theta \circ ([\Theta,\Theta] \vee [\Psi,\Psi]) \circ \Psi \subseteq \\
&\subseteq \Theta \circ ([\Theta,1] \vee [\Psi,1]) \circ \Psi \subseteq \\
&\subseteq \Theta \circ ([\Theta,\pi_1 \vee \pi_2] \vee [\Psi,\pi_1 \vee \pi_2]) \circ \Psi \subseteq \\
&\subseteq \Theta \circ ([\Theta,\pi_1] \vee [\Theta,\pi_2] \vee [\Psi,\pi_1] \vee [\Psi,\pi_2]) \circ \Psi \subseteq \\
&\subseteq \Theta \circ ((\Theta \wedge \pi_1) \vee (\Psi \wedge \pi_1) \vee (\Theta \wedge \pi_2) \vee (\Psi \wedge \pi_2)) \circ \Psi \subseteq \\
&\subseteq \Theta \circ (((\Theta \wedge \pi_1) \circ (\Psi \wedge \pi_1)) \vee ((\Theta \wedge \pi_2) \circ (\Psi \wedge \pi_2))) \circ \Psi \subseteq \\
&\subseteq \Theta \circ ((\Theta \wedge \pi_1) \circ (\Psi \wedge \pi_1) \circ (\Theta \wedge \pi_2) \circ (\Psi \wedge \pi_2)) \circ \Psi \subseteq \\
&\subseteq \Theta \circ (\Psi \wedge \pi_1) \circ (\Theta \wedge \pi_2) \circ \Psi \subseteq \\
&\subseteq \Theta \circ (\Theta \circ \pi_2) \circ (\Psi \circ \pi_1) \circ \Psi \subseteq \underline{\Theta \circ \Psi}
\end{aligned}$$

We have twice used the fact that every congruence below π_1 permutes with every congruence below π_2. This follows from 4.4 because π_1 and π_2 permute.

Quite useful is the following instance of Corollary 8.2:

8.9 <u>Corollary:</u> If $\mathrm{Con}(\underline{A})$ has M_3, (the five-element modular nondistributive lattice) as a sublattice, then any two elements of this sublattice permute.

<u>Proof:</u> Clearly if



is a sublattice, then $[\alpha,\alpha] \leq \varepsilon$ by successively applying the rules of 6.16. Hence $\alpha, \beta, \gamma, \delta$ are solvable in $\underline{A}/\varepsilon$. Thus in fact $\alpha, \beta, \gamma, \delta$ permute with everything above ε.

As an example, using arguments from PIXLEY [35], a result of R. McKENZIE [33] can be adapted to the modular case:

8.10 <u>Corollary:</u> If $\underline{A}$ is finite, simple in a modular variety, then $\underline{A}$ is either affine or $\underline{A}^+$ generates a distributive variety. (Here $\underline{A}^+$ is the algebra obtained from $\underline{A}$ by adding a constant $\underline{a}$ with value a for every element $a \in A$.)

<u>Proof:</u> Let $\underline{F} := \underline{F}_{V(\underline{A}^+)}(3)$ be freely three-generated in the variety $V(\underline{A}^+)$ generated by A^+. Clearly $\underline{F}$ is a subdirect power of $\underline{A}^+$. If $Con(\underline{F})$ is distributive, then $V(\underline{A}^+)$ is congruence distributive as a consequence of JONSSON's theorem (thm. 1.4). Otherwise there are coatoms α, β in $Con(\underline{F})$ with $\{x \mid \alpha \wedge \beta \leq x \leq 1\}$ being a nondistributive interval; α and β may be chosen to be part of the subdirect decomposition for $\underline{F}$ (see BURRIS [5]). Since $\underline{F}/\alpha \cong \underline{F}/\beta \cong \underline{A}^+$, and, by corollary 8.9, $\underline{F}/\alpha\wedge\beta \cong \underline{A}^+ \times \underline{A}^+$ for $\alpha \neq \beta$. Hence $\underline{A}^+ \times \underline{A}^+$ has a congruence which is a complement of the canonical factor congruences, since $\underline{A}^+$ is simple.

9. ABELIAN CONGRUENCES AND AFFINE ALGEBRAS

In the preceding chapters we have mainly made use of the commutators of the form $[\alpha,\alpha]$. Here we look at a case, slightly more general, namely we assume $\alpha \geq \beta$ and consider $[\alpha,\beta]$.

If x, y, z are elements with $x\ \alpha\ y\ \beta\ z$ then we have already found the important relation

$$(x,y)\ \Delta_{\alpha}^{\beta}\ (t(x,y,z),z). \qquad (\S)$$

Therefore, since Δ_{α}^{β} is a congruence we find for $\vec{x} := (x_1,\dots,x_n)$, $\vec{y} := (y_1,\dots,y_n)$, $\vec{z} := (z_1,\dots,z_n)$ with $x_i\ \alpha\ y_i\ \beta\ z_i$ and any n-ary operation f:

$$(f(\vec{x}),f(\vec{y}))\ \Delta_{\alpha}^{\beta}\ (f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n)),f(\vec{z})).$$

Using (§) again we get

$$(f(\vec{x}),f(\vec{y}))\ \Delta_{\alpha}^{\beta}\ (t(f(\vec{x}),f(\vec{y}),f(\vec{z})),f(\vec{z}))$$

and hence

$$f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n))\ [\alpha,\beta]\ t(f(\vec{x}),f(\vec{y}),f(\vec{z})). \qquad (\S\S)$$

This yields one direction of an equational description of $[\alpha,\beta]$:

9.1 <u>Theorem:</u> Suppose $\alpha \geq \beta$. Then $[\alpha,\beta] = 0$ if and only if for all $x_i\ \alpha\ y_i\ \beta\ z_i$ with $x_i, y_i, z_i \in A$ the equations

$$t(y_i,y_i,z_i) = z_i \quad \text{and}$$

$$f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n)) = t(f(\vec{x}),f(\vec{y}),f(\vec{z}))$$

are satisfied.

<u>Proof:</u> For the proof of the missing direction we define a congruence relation $\equiv$ on α by

$$(x,y) \equiv (u,z) :\Longleftrightarrow x\ \alpha\ y\ \beta\ z \quad \text{and} \quad t(x,y,z) = u.$$

To show symmetry we suppose $t(x,y,z) = u$ and $x\ \alpha\ y\ \beta\ z$ and compute:

$$\begin{aligned} t(u,z,y) &= t(t(x,y,z),t(y,y,z),t(y,y,y)) = \\ &= t(t(x,y,y),t(y,y,y),t(z,z,y)) = \\ &= t(x,y,y) = x. \end{aligned}$$

Clearly $x\ \beta\ u$, hence $u\ \alpha\ z\ \beta\ y$.

For transitivity suppose $x\ \alpha\ y\ \beta\ z$, $u\ \alpha\ z\ \beta\ s$, $t(x,y,z) = u$, $t(u,z,s) = r$ and compute:

$$\begin{aligned} t(x,y,s) &= t(t(x,y,y),t(y,y,y),t(z,z,s)) = \\ &= t(t(x,y,z),t(y,y,z),t(y,y,s)) = \\ &= t(u,z,s) = r. \end{aligned}$$

Again $x\ \alpha\ y\ \beta\ s$ trivially.

Using that $t(x,x,y) = y$ for $x\ \beta\ y$ we find $(x,x) \equiv (y,y)$ whenever $x\ \beta\ y$ and consequently $\equiv\ \geq \Delta_\alpha^\beta$.

Hence suppose $x\ [\alpha,\beta]\ y$, then $(y,x)\ \Delta_\alpha^\beta\ (x,x)$, therefore $(y,x) \equiv (x,x)$ hence $t(y,x,x) = x$ which implies $x = y$. Thus $[\alpha,\beta] = 0$.

We have created a situation similar to the hypothesis of Proposition 5.4. Indeed, theorem 9.1 permits us to associate affine algebras with congruences α which are abelian, i.e. for which $[\alpha,\alpha] = 0$. If additionally α is contained in the center, then all affine algebras associated with α are isomorphic. Moreover, the congruences below the center ξ correspond uniquely to the subalgebras of the affine algebra associated with ξ. This reflects the group theoretic situation where subalgebras of the center of $\underline{G}$ are normal in $\underline{G}$. One of the difficulties we have here is, that our algebras need not have any one-element subalgebras.

We have to use 9.1 over and over again. The approach is rather naturally and intuitively clear, but we have to be careful in setting up the right equations so that the conditions in 9.1 remain satisfied.

Choose an arbitrary element a from $\underline{A}$ and a congruence relation $\beta \in \mathrm{Con}(\underline{A})$. Let f be a fundamental operation of $\underline{A}$, or the ternary operation $t(x,y,z)$. For $x_1,\dots,x_n \in [a]\beta$ we define

$$f^\nabla(x_1,\dots,x_n) := t(a,f(a,\dots,a),f(x_1,\dots,x_n)).$$

Let $\underline{A}^\nabla[\beta]_a$ be the algebra with base set $[a]\beta$ and the operations of the form f^∇. Clearly $[a]\beta$ is closed under the new operations so that the definition makes sense. Note that in case $[\beta,\beta] = 0$ idempotent operations remain unchanged, in particular, if $\{a\}$ happened to be a one-element subalgebra of $\underline{A}$ then $\underline{A}^\nabla[\beta]_a$ is the subalgebra $[a]\beta$ of $\underline{A}$. This is immediate from 7.1.

First we need:

9.2 <u>Theorem:</u> Let $\alpha \geq \beta$ with $[\alpha,\beta] = 0$. Then $\underline{A}^\nabla[\beta]_a$ is an affine algebra and for $(a,b) \in \alpha$ we get $\underline{A}^\nabla[\beta]_a \cong \underline{A}^\nabla[\beta]_b$.

9.3 <u>Corollary:</u> If $\alpha \geq \beta$ and $[\alpha,\beta] = 0$ then β is uniform with respect to α, i.e. all β-classes within a fixed α-class have the same size.

Defining $\alpha^1 := \alpha$ and $\alpha^{n+1} := [\alpha^n, 1]$ then by induction:

9.4 Corollary: If $\alpha^n = 0$ for some $n \in \mathbb{N}$ then all classes of α have the same size (α is a uniform congruence).

Proof of 9.2: a) $\underline{A}^\nabla[\beta]_a$ is affine with respect to t $(= t^\nabla)$. We use the notation $\overline{a}$ for the constant sequence $(a,\dots,a)$. $\vec{x}$ denotes $(x_1,\dots,x_n)$, similarly we use $\vec{y}$ and $\vec{z}$.
So, given $x_i, y_i, z_i \in [a]\beta$ we compute

$$\begin{aligned} t(f^\nabla(\vec{x}),f^\nabla(\vec{y}),f^\nabla(\vec{z})) &= t(t(a,f(\overline{a}),f(\vec{x})),t(a,f(\overline{a}),f(\vec{y})),t(a,f(\overline{a}),f(\vec{z}))) \\ &= t(t(a,a,a),t(f(\overline{a}),f(\overline{a}),f(\overline{a})),t(f(\vec{x}),f(\vec{y}),f(\vec{z}))) \\ &= t(a,f(\overline{a}),f(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n))) \\ &= f^\nabla(t(x_1,y_1,z_1),\dots,t(x_n,y_n,z_n)). \end{aligned}$$

b) For $a \,\alpha\, b$ we show $\underline{A}^\nabla[\beta]_a \cong \underline{A}^\nabla[\beta]_b$:
Define $\xi_{a,b}\colon [a]\beta \twoheadrightarrow [b]\beta$ by $\xi_{a,b}(x) := t(b,a,x)$, then with 9.1 again

$$\begin{aligned} \xi_{b,a}\circ\xi_{a,b}(x) &= t(a,b,t(b,a,x)) = t(t(a,a,a),t(b,a,a),t(b,a,x)) = \\ &= t(t(a,b,b),t(a,a,a),t(a,a,x)) = t(a,a,x) = x. \end{aligned}$$

Hence the $\xi_{a,b}$ are bijective for $a \,\alpha\, b$.
Furthermore, (denoting the operations of $\underline{A}^\nabla[\beta]_a$ and of $\underline{A}^\nabla[\beta]_b$ with the same symbol) we get for $x_i \in [a]\beta$:

$$\begin{aligned} &\xi_{a,b}(f^\nabla(x_1,\dots,x_n)) = t(b,a,t(a,f(\overline{a}),f(\vec{x}))) = \\ &= t(t(b,f(\overline{b}),f(\overline{b})),t(a,f(\overline{a}),f(\overline{a})),t(a,f(\overline{a}),f(\vec{x}))) = \\ &= t(t(b,a,a),t(f(\overline{b}),f(\overline{a}),f(\overline{a})),t(f(\overline{b}),f(\overline{a}),f(\vec{x}))) = \\ &= t(b,f(\overline{b}),f(t(b,a,x_1),\dots,t(b,a,x_n))) \\ &= f^\nabla(\xi_{a,b}(x_1),\dots,\xi_{a,b}(x_n)). \end{aligned}$$

Now from the above it is clear that in the special case where $[1,\alpha] = 0$, i.e. $\alpha \le \zeta$ (the center of $\underline{A}$) there is one affine algebra $\underline{A}[\alpha]$ associated with α.
$\underline{A}[\alpha]$ even is contained in the variety generated by $\underline{A}$ and the relation between $\underline{A}$, $\underline{A}[\alpha]$ and (the algebra) $\underline{\alpha}$ is given by

9.5 Theorem: If $[1,\alpha] = 0$ then

(i) $A^\nabla[\alpha] \cong \underline{\alpha}/\Delta^1_\alpha$ and

(ii) $\underline{\alpha} \cong \underline{A} \times A^\nabla[\alpha]$.

9.5(ii) is a generalization of the situation which has first appeared in Sec. 5, namely for A an affine algebra, i.e. setting $\alpha = 1$ we get $\underline{A} \times \underline{A} = \underline{A} \times \underline{A}^\nabla$ where now $\underline{A}^\nabla$ (by 9.5(i)) is the factor of $\underline{A} \times \underline{A}$ by the congruence $\Delta = \Delta_1^1$ on $\underline{A}\times\underline{A}$ (see Sec. 5). Note that $\underline{A}^\nabla[\alpha]$ always has a one-element subalgebra, as (possibly) opposed to $\underline{A}$.

Proof of 9.5: (i) Define a map $\delta_a: \underline{\alpha} \twoheadrightarrow [a]\alpha$ by $\delta_a(x,y) := t(a,x,y)$, then δ_a is a map from α to $[a]\alpha$ and $\ker \delta_a = \Delta_\alpha^1$ because suppose $t(a,x,y) = t(a,x',y') =: u$ then $(a,x)\ \Delta_1^\alpha\ (u,y)$ and $(a,x')\ \Delta_1^\alpha\ (u,y')$ so $(a,u)\ \Delta_\alpha^1\ (x,y)$ and $(a,u)\ \Delta_\alpha^1\ (x',y')$ according to the beginning of this section, hence $(x,y)\ \Delta_\alpha^1\ (x',y')$, i.e. $\ker \delta_a \le \Delta_\alpha^1$. Conversely $(x,y)\ \Delta_\alpha^1\ (x',y')$ leads to $(x,x')\ \Delta_1^\alpha\ (y,y')$ and for $v := t(a,x,y)$ we find $(a,x)\ \Delta_1^\alpha\ (v,y)$ hence $(a,x')\ \Delta_1^\alpha\ (v,y')$ and $t(a,x,y) = v = t(a,x',y')$.

For the homomorphism condition we calculate

$$
\begin{aligned}
\delta_a(f((x_1,y_1),\ldots,(x_n,y_n))) &= t(a,f(\vec{x}),f(\vec{y})) = \\
&= t(a,a,t(a,f(\vec{x}),f(\vec{y}))) \\
&= t(t(a,a,a),t(a,f(\bar{a}),f(\bar{a})),t(a,f(\vec{x}),f(\vec{y}))) \\
&= t(a,f^\nabla(\vec{x}),f^\nabla(\vec{y})) \\
&= f^\nabla(t(a,x_1,y_1),\ldots,t(a,x_n,y_n)) \\
&= f^\nabla(\delta_a(x_1,y_1),\ldots,\delta_a(x_n,y_n)).
\end{aligned}
$$

For the proof of (ii) we find in the congruence lattice of the algebra $\underline{\alpha}$ the congruences Δ_α^1, π_1 and π_2 (the kernels of the projections onto A). $[1,\alpha] = 0$ means $\Delta_\alpha^1 \wedge \pi_1 = 0$ and, equivalently $\Delta_\alpha^1 \wedge \pi_2 = 0$. Moreover $\Delta_\alpha^1 \vee \pi_1 = \Delta_\alpha^1 \vee \pi_2 = 1$. Thus Δ_α^1 and π_1 are complements in $\mathrm{Con}(\underline{\alpha})$. If they permute they will give the desired decomposition of $\underline{\alpha}$. But since $[\pi_1,\pi_1] = 0$ follows from the above relations, π_1 permutes with every congruence (recall 8.2).

Next we are going to look at congruences below the center ζ and we shall show that they correspond uniquely to subalgebras of $\underline{A}[\zeta]$.

Fix an element a from $\underline{A}$. For a subalgebra $\underline{S}$ of $\underline{A}^\nabla[\zeta]$ containing the element a, define $\Psi(\underline{S}) := \{(x,y) \in \zeta \mid t(a,x,y) \in \underline{S}\}$ and for a congruence relation Θ of $\underline{A}$ with $\Theta \le \zeta$ define $U(\Theta) := [a]\Theta$. We show

9.6 Proposition: $U(-)$ and $\Psi(-)$ are mutually inverse lattice isomorphisms between the interval $[0,\zeta]$ of $\mathrm{Con}(\underline{A})$ and $\mathrm{Sub}(\underline{A}^\nabla[\zeta])$, the lattice of subalgebras of $\underline{A}^\nabla[\zeta]$.

Remark: The point is here that subalgebras of $\underline{A}^\nabla[\zeta]$ correspond to congruences on $\underline{A}$. For congruences on $\underline{A}^\nabla[\zeta]$ the corresponding property is trivial from affineness. Another way to phrase 9.6 would be: Congruences on $\underline{A}^\nabla[\zeta]$ can be extended to congruences on $\underline{A}$.

Proof: We show that $\Psi(\underline{S})$ is a congruence on $\underline{A}$.

Symmetry: If $x\ \Psi(\underline{S})\ y$ then $a\ \zeta\ t(a,y,x)$ hence

$$\begin{aligned} t(a,y,x) &= t(a,a,t(a,y,x)) \\ &= t(t(a,a,a),t(a,x,x),t(a,y,x)) \\ &= t(a,t(a,x,y),a) \in \underline{S}. \end{aligned}$$

Transitivity: $t(a,x,y) \in S$ and $t(a,y,z) \in \underline{S}$, $x\zeta y\zeta z$ imply

$$\begin{aligned} t(a,x,z) &= t(t(a,a,a),t(x,y,y),t(y,y,z)) \\ &= t(t(a,x,y),a,t(a,y,z)) \in \underline{S}. \end{aligned}$$

Compatibility: Given $x_i\zeta y_i$ and $t(a,x_i,y_i) \in \underline{S}$ then $f(\vec{x})\zeta f(\vec{y})$ and

$$\begin{aligned} & t(a,f(\vec{x}),f(\vec{y})) = \\ &= t(t(a,f(\bar{a}),f(\bar{a})),t(f(\vec{x}),f(\vec{x}),f(\vec{x})),t(f(\vec{x}),f(\vec{x}),f(\vec{y}))) \\ &= t(a,f(\bar{a}),f(t(a,x_1,y_1),\dots,t(a,x_n,y_n))) \\ &= f^{\nabla}(t(a,x_1,y_1),\dots,t(a,x_n,y_n)) \in \underline{S}. \end{aligned}$$

Finally for $\Theta \leq \zeta$, $\Theta \in \mathrm{Con}(\underline{A})$:
$x\ \Psi\ U(\Theta)\ y$ <u>iff</u> $t(a,x,y) \in U(\Theta)$ & $x\zeta y$ <u>iff</u>
$t(a,x,y)\ \Theta\ t(a,x,x) = a$ & $x\zeta y$ <u>iff</u> $x\Theta y$.

The last equivalence here is due to

$$\begin{aligned} x = t(x,a,a)\ \Theta\ t(x,a,t(a,x,y)) &= t(t(x,x,x),t(a,x,x),t(a,x,y)) \\ &= t(x,x,y) = y. \end{aligned}$$

With the preceding results the congruences below the center and the corresponding affine algebras seem to be well understood. Next abelian congruences β i.e. those for which $[\beta,\beta] = 0$ should be studied for $\beta \nleq \xi$. In general the corresponding affine algebras $\underline{A}^{\nabla}[\beta]_a$ do not lie in the variety generated by $\underline{A}$. In the case of groups though they do generate equivalent varieties. A description of those affine algebras might lead to improving the bounds for the cardinality of subdirectly irreducible algebras in residually finite varieties, given by FREESE, McKENZIE [13].

10. VARIETIES OF AFFINE ALGEBRAS

Analogously to the theory of groups we define for a congruence α:

$$\alpha_{(o)} := \alpha; \qquad \alpha^{(o)} := \alpha$$

$$\alpha_{(n+1)} := [\alpha,\alpha_{(n)}]; \qquad \alpha^{(n+1)} := [\alpha^{(n)},\alpha^{(n)}].$$

Then α is called <u>nilpotent</u> (solvable) of degree $\le k$ if $\alpha_{(k)} = 0$ ($\alpha^{(k)} = 0$). α is called nilpotent (solvable) if for some $k \in \mathbb{N}$ α is nilpotent (solvable) of degree $\le k$. Clearly $\underline{A}$ is called nilpotent (solvable) if $1_{\underline{A}}$ is.

If $\underline{V}$ is a variety then with $\underline{V}_{(k)}$ (resp. $\underline{V}^{(k)}$) we denote the class of all algebras which are nilpotent (resp. solvable) of degree $\le k$.

Starting with $t(x,y,z)$ from chapter 4 (which may of course be $p(z,y,x)$ from chapter 7) we define recursively:

$$t_1(x,y,z) := t(x,y,z)$$

$$t_{n+1}(x,y,z) := t(t_n(x,y,z),t_n(y,y,z),z).$$

Then we get:

10.1 <u>Observation:</u> $\underline{V}_{(k)}$ and $\underline{V}^{(k)}$ are permutable varieties with Mal'cev term t_k.

<u>Proof:</u> The fact that $\underline{V}_{(k)}$ and $\underline{V}^{(k)}$ are varieties can be proven just as in the case of groups. In other words, the class of algebras with a solvable (nilpotent) series of length $\le k$ for some fixed k, is closed under taking homomorphic images, subalgebras and direct products. It has been clear from 8.2, that $\underline{V}_{(k)}$ and $\underline{V}^{(k)}$ are permutable varieties. The fact that t_k is a Mal'cev term on every solvable algebra (degree $\le k$) is easy by induction using 7.1.

Now we are going to use 9.1 to give an equational description for $\underline{V}_{(k)}$ and for $\underline{V}^{(k)}$.

To this end let us define by recursion:

$$N_o := \{x \equiv y\}$$

$$N_{k+1} := \{t(\sigma,\sigma,\tau) \equiv \tau \mid N_k \vdash \sigma \equiv \tau\} \cup$$

$$\cup \{f(t(x_1,\sigma_1,\tau_1),\dots,t(x_n,\sigma_n,\tau_n)) \equiv$$

$$\equiv t(f(x_1,\dots,x_n),f(\sigma_1,\dots,\sigma_n),f(\tau_1,\dots,\tau_n)) \mid$$

$$\mid f \text{ is n-ary operation and } N_k \vdash \sigma_i \equiv \tau_i \text{ for } 0 \le i \le n\}$$

and

$$S_o := \{x \equiv y\}$$

$$S_{k+1} := \{t(\sigma,\sigma,\tau) \equiv \tau \mid S_k \vdash \sigma \equiv \tau\} \cup$$
$$\cup \{f(t(\gamma_1,\sigma_1,\tau_1),\dots,t(\gamma_n,\sigma_n,\tau_n)) \equiv$$
$$\equiv t(f(\gamma_1,\dots,\gamma_n),f(\sigma_1,\dots,\sigma_n),f(\tau_1,\dots,\tau_n)) \mid$$
$$\mid f \text{ n-ary operation and } S_k \vdash \gamma_i \equiv \sigma_i \equiv \tau_i,\ \ 0 \le i \le n\}.$$

9.2 yields by induction:

10.2 Corollary: Relatively to $\underline{V}$ the varieties $\underline{V}_{(k)}$ (resp. $\underline{V}^{(k)}$) are defined by the equation N_k (resp. S_k).

For the case $k = 1$ nilpotency and solvability coincide, leaving us with the variety of affine algebras.

We have seen before that affine algebras are just modules in disguise. Indeed $\underline{V}^{(1)}$ is polynomially equivalent to a variety of modules over a ring $R(\underline{V})$. Let us have a closer look at $R(\underline{V})$.

Clearly we should expect $R(\underline{V})$ to be the free module on one generator. However, instead of looking at $\underline{F}_{\underline{V}}(1)$, where we have to make up our mind which element to pick for a o-element, we just adjoin a new free generator, which we call o. Thus technically we look at the free $\underline{V}$-algebra with free generators x and o. (This idea goes back at least to CSAKANY [7].) Then we do the construction of § 5, i.e. $R(\underline{V})$ has an underlying set the set of all binary idempotent terms $r(o,x)$. Addition and multiplication are given by

$$r(o,x) + s(o,x) := t(r(o,x),o,s(o,x))$$
$$r(o,x) \cdot s(o,x) := r(o,s(o,x)).$$

Clearly, since $[1,1] = 0$, by chapter 5 we have defined an abelian group with an associative multiplication. The one distributive law which is nontrivial, is a consequence of 9.1.

Obviously, for every $\underline{A} \in \underline{V}$, the ring $R(\underline{A})$ from § 5 is a homomorphic image of $R(\underline{V})$.

11. GENERALIZATIONS: FP-VARIETIES

The crucial idea that eventually led us to the consideration of commutators was the idea of coordinatizing the congruence class geometry. The important results about modular varieties which were needed were:

1. that congruences on direct products permute with the factor congruences

and

2. the Shifting Lemma.

Clearly, as we have seen in Corollary 3.6, the Shifting Lemma in its general form is equivalent to modularity. The first property however, is strictly weaker. Thus let us define:

11.1 <u>Definition:</u> A variety $\underline{V}$ is called <u>factor permutable</u> (or in short: FP-variety) if every congruence relation on a direct product $\underline{A} \times \underline{B}$ of algebras $\underline{A},\underline{B} \in \underline{V}$ permutes with the canonical factor congruences π_1 and π_2.

Recall that in general a factor congruence α is a congruence for which there exists another congruence β with $\alpha \wedge \beta = 0$ and $\alpha \circ \beta = 1$.

As an example let us look at the variety $\underline{W}$ given by one ternary term $p(x,y,z)$ and a constant 0, satisfying the equations

$$p(x,x,y) = y$$
$$p(x,0,0) = x.$$

Then $\underline{W}$ is <u>not</u> a modular variety, indeed no equation is satisfied in all congruences lattices of algebras in $\underline{W}$. To witness, let S be any nonempty set and $0 \notin S$. On $A := S \cup \{0\}$ define the ternary operation $p(x,y,z)$ by

$$p(x,y,z) := \begin{cases} x & \text{if } y = z = 0 \\ z & \text{else.} \end{cases}$$

Then clearly we obtain an algebra in $\underline{W}$. Moreover, every partition of S together with the singleton class $\{0\}$, is a congruence relation of $\underline{A}$.

On the other hand, to see that $\underline{W}$ is an FP-variety, let a congruence relation Θ be given on $\underline{A} \times \underline{B} \in \underline{W}$. Suppose $x\ \pi_1\ y\ \Theta\ z$, then for some appropriate elements $x = (a_1,b_1)$, $y = (a_1,b_2)$ and $z = (a_2,b_3)$. It follows

$$p((a_1,b_2),(0,b_2),(0,b_1))\ \Theta\ p((a_2,b_3),(0,b_2),(0,b_1))$$

hence

$$(a_1,b_1)\ \Theta\ (a_2,p(b_3,b_2,b_1)),$$

thus with $u := (a_2,p(b_3,b_2,b_1))$ we get $x\ \Theta\ u\ \pi_1\ z$.

Other examples of FP-varieties which are not already modular include those varieties studied by FRASER and HORN [10] and by HU [27]. Those are varieties where congruences on direct products are products of congruences on the factors.

A simple example from [10] is defined by the equations $x + 0 = 0 + x = x \cdot 1 = x$, $x \cdot 0 = 0$. FRASER and HORN have given a Mal'cev-type description of their varieties. A similar description of FP-varieties is as easy:

11.2 Theorem: A variety $\underline{W}$ is an FP-variety, if and only if there exist natural numbers $m,n \geq 1$, a map $k: \{1,\dots,n\} \to \{0,1\}$, $(m+1)$-ary terms $p_1,\dots,p_n$, binary terms r_{ij} and ternary terms s_{ij} with $1 \leq i \leq n$, $1 \leq j \leq m$ such that the following equations hold in $\underline{W}$:

(1) $x_0 = p_1(x_{k(1)},r_{11},\dots,r_{1m})$

(2) $x_1 = p_n(x_{1-k(n)},r_{n1},\dots,r_{nm})$

(3) $x_2 = p_n(x_{1-k(n)},s_{n1},\dots,s_{nm})$

(4) $p_i(x_{1-k(i)},r_{i1},\dots,r_{im}) = p_{i+1}(x_{k(i+1)},r_{i+1,1},\dots,r_{i+1,m})$ for all $i < n$

(5) $p_i(x_{1-k(i)},s_{i1},\dots,s_{im}) = p_{i+1}(x_{k(i+1)},s_{i+1,1},\dots,s_{i+1,m})$ for all $i < n$.

Proof: The proof is by looking at the congruence relation Θ on the direct product $\underline{F}_W(\{x_0,x_1\}) \times \underline{F}_W(\{x_0,x_1,x_2\})$ generated by the pair $((x_0,x_0),(x_1,x_1))$. The arguments are routine, see [23].

Note that adding the equation

$$x_2 = p_1(x_{k(1)},s_{n1},\dots,s_{nm})$$

to the above equations yields the condition of FRASER and HORN.

Remark: The case $n = 1$ is particularly interesting. If $k(1) = 1$ then $\underline{W}$ is a permutable variety with $p(x,r_1(y,z),\dots,r_m(y,z))$ being the MAL'CEV term.

If $k(1) = 0$ and $m = 1$ we are left with the equations

(Σ1) $x_0 = p(x_0,r(x_0,x_1))$

(Σ2) $x_1 = p(x_1,r(x_0,x_1))$

(Σ3) $x_2 = p(x_1,s(x_0,x_1,x_2))$

11.3 <u>Proposition:</u> The equations $\Sigma 1$, $\Sigma 2$, $\Sigma 3$ jointly imply that every finite algebra in $\underline{W}$ generates a permutable variety.

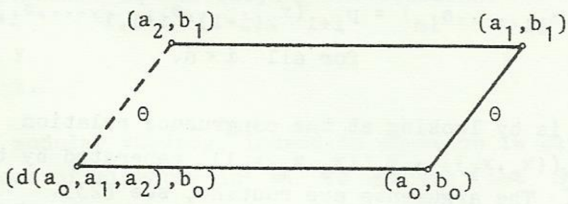
<u>Proof:</u> $\Sigma 3$ implies that the map $p(a,-)$ is onto for every $a \in \underline{A}$. If A is finite then $p(a,-)$ must be 1-1. Equations $\Sigma 1$ and $\Sigma 2$ yield $p(x,r(x,y)) = p(x,r(y,x))$, hence $r(x,y) = r(y,x)$ be the above. $\Sigma 1$ alone yields $p(x,r(x,y)) = p(x,r(x,z))$ hence $r(x,y) = r(x,z)$. Combining this, $r(x,y) = r(x,z) = r(z,x) = r(z,u)$ thus $r(x,y)$ is a constant, $0 := r(x,y)$.
Similarly, $\Sigma 3$ gives $s(x_o,x_1,x_2) = s(y,x_1,x_2)$ hence s does not depend on the first place, i.e. $s(x_o,x_1,x_2) =: u(x_1,x_2)$. Furthermore $p(x_o,0) = x_o = p(x_o,u(x_o,x_o))$ so $u(x_o,x_o) = 0$. Combining this we define

$$d(x_o,x_1,x_2) := p(x_o,u(x_1,x_2)),$$

then the above arguments show that d is a Mal'cev term on every finite algebra.

The term $d(x,y,z)$ as constructed above plays a central rôle in FP-varieties:

11.4 <u>Proposition:</u> In every FP-variety $\underline{W}$ there exists a term $d(x,y,z)$ such that $d(x,x,y) = y$ is an equation in $\underline{W}$ and for every congruence relation θ on $\underline{A} \times \underline{B}$ with $(a_o,b_o)\ \theta\ (a_1,b_1)$ and for every $a_2 \in \underline{A}$, $(d(a_o,a_1,a_2),b_o)$ completes the following parallelogram:



<u>Proof:</u> Depending on whether $k(1)$ is 0 or 1 in the Mal'cev condition we define

$$d(x,y,z) := p_1(x,s_{11}(x,y,z),\ldots,s_{1m}(x,y,z))$$

or

$$d(x,y,z) := p_1(y,s_{11}(x,y,z),\ldots,s_{1m}(x,y,z)).$$

Now for $1 \le i \le n$, $j \in \{0,1\}$ we set

$$s_i^j := p_i(a_j,s_{i1}(a_o,a_1,a_2),\ldots,s_{im}(a_o,a_1,a_2))$$

and

$$t_i^j := p_i(b_j,r_{i1}(b_o,b_1),\ldots,r_{im}(b_o,b_1)).$$

Then $(s_i^j, t_i^j)\ \Theta\ (s_i^k, t_i^k)$ for $j,k \in \{0,1\}$ and

$d(a_o,a_1,a_2),b_o) = (s_1^{k(1)}, t_1^{k(1)})$ by equation (1),

$(s_i^{1-k(i)}, t_i^{1-k(i)}) = (s_{i+1}^{k(i+1)}, t_{i+1}^{k(i+1)})$ by equations (5) and (4) and

$(s_n^{1-k(n)}, t_n^{1-k(n)}) = (a_2,b_1)$ by equations (3) and (2). Thus

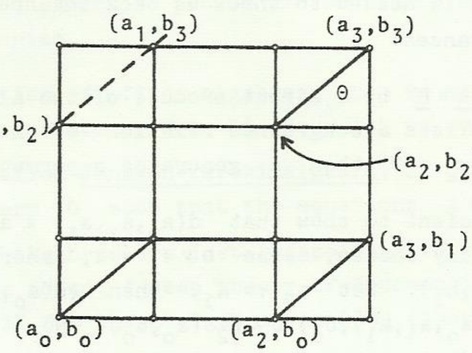
$(d(a_o,a_1,a_2),b_o)\ \Theta\ (a_2,b_1)$ by transitivity.

To see that $d(x,x,y) = y$, set $\Theta := \pi_1$ on $\underline{A} \times \underline{A}$ and $b_1 = a_1 = a_o = x$ and $a_2 = b_o = y$. Then the above implies that
$d(x,x,y) = d(a_o,a_1,a_2) = a_2 = y$.

We do also have a weak replacement for the Shifting Lemma in an FB-variety, i.e. it is clear that the Shifting Lemma 2.1 holds, provided β is a factor congruence.

As we have seen in the examples, we cannot expect the congruence class geometry of algebras in FP-varieties to behave nicely on every algebra. However, direct products are still comparatively well behaved, and counterparts for closure theorems in the modular case can be found. Thus e.g. the Cube lemma will have to be replaced by

11.5 The REIDEMEISTER-theorem: Let Θ be a congruence relation on $\underline{A} \times \underline{B}$ with $\Theta \wedge \pi_2 = 0$. Then

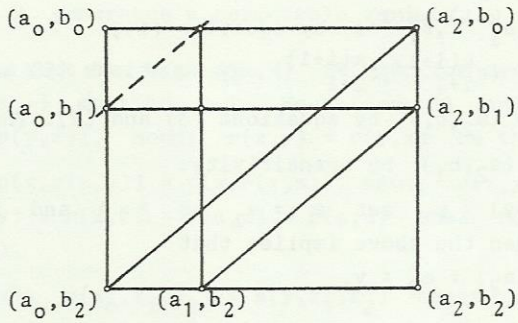


implies $(a_o,b_2)\ \Theta\ (a_1,b_3)$.

Proof: According to 11.4 we have $(d(a_2,a_3,a_1),b_o)\ \Theta\ (a_1,b_1)$ hence $(d(a_2,a_3,a_1),b_o)\ \Theta \wedge \pi_2\ (a_o,b_o)$ whence $d(a_2,a_3,a_1) = a_o$.
It follows then from $(d(a_2,a_3,a_1),b_2)\ \Theta\ (a_1,b_3)$ that $(a_o,b_2)\ \Theta\ (a_1,b_3)$ which we claimed.

There is also a pendant to the Escher Cube which we will need later:

11.6 Lemma: Let Θ be a congruence relation on the direct product $\underline{A} \times \underline{B}$. If $(a_2,b_o)\ \Theta\ (a_o,b_2)$, $(a_1,b_2)\ \Theta\ (a_2,b_1)$ then $(a_1,b_o)\ \Theta\ (a_o,b_1)$.



Proof: Factor permutability accounts for the existence of a_3, b_3 with $(a_3,b_1)\ \theta\ (a_1,b_3)\ \theta\ (a_o,b_2)$. We apply d from 11.4 to obtain

$d((a_3,b_1),(a_2,b_1),(a_1,b_1)) = (d(a_3,a_2,a_1),b_1)$ and

$d((a_1,b_3),(a_1,b_2),(a_1,b_1)) = (a_1,d(b_3,b_2,b_1))$.

Since corresponding entries in d are filled with congruent elements, we get, using the geometrical properties of d from 11.4:

$$(a_1,b_o)\ \theta\ (d(a_3,a_2,a_1),b_1)\ \theta\ (a_1,d(b_3,b_2,b_1))\ \theta\ (a_o,b_1).$$

The following lemma is needed to throw us back into permutable varieties in certain circumstances:

11.7 Lemma: Let $\underline{A}\times\underline{B}$ be a direct product of two algebras in an FP-variety. If there exists a congruence relation θ on $\underline{A}\times\underline{B}$ with $\theta\vee\pi_1 = 1$ and $\theta\wedge\pi_2 = 0$ then $\underline{A}$ generates a permutable variety.

Proof: It is sufficient to show that $d(a_o,a_1,a_1) = a_o$ holds for $a_o,a_1\in\underline{A}$ arbitrarily chosen. Since $\theta\vee\pi_1 = 1$, there exist $b_o,b_1\in\underline{B}$ with $(a_o,b_o)\ \theta\ (a_1,b_1)$. Set $a_2 := a_1$, then $(d(a_o,a_1,a_1),b_o)\ \theta\ (a_1,b_1)$ by 11.4, hence $(d(a_o,a_1,a_1),b_o)\ \theta\wedge\pi_2\ (a_o,b_o)$. Now the condition $\theta\wedge\pi_2 = 0$ forces $d(a_o,a_1,a_1) = a_o$.

Thus the characterization theorem for affine algebras in permutable varieties from [16] carries over unchanged to FP-varieties. The commutator machinery was used in [26] to carry the original result ([16], Theorem 4.7) over to modular varieties. That it could be done without was shown in [17]. The above lemma shows that the original proof still works in the FP case.

11.8 Theorem: Let $\underline{A}$ be an algebra in an FP-variety. Then the following conditions are equivalent:

(i) $\underline{A}$ is affine

(ii) There exists a congruence relation θ on $\underline{A}\times\underline{A}$ which is a common complement of π_1 and of π_2.

(iii) $\mathrm{diag}(\underline{A}) = \{(x,x) \mid x \in \underline{A}\}$ is a congruence class on $\underline{A} \times \underline{A}$.

Proof: Only (iii) → (ii) needs a proof. The rest follows with 11.7 and chapter 5.

Let Ψ be the congruence relation of which $\mathrm{diag}(\underline{A})$ is a class. $\Psi \vee \pi_i = 1$ is clear for $i \in \{0,1\}$. If $\Psi \wedge \pi_1$, say, is different from 0, i.e. $(a,b)\ \Psi\ (a,c)$ then the Shifting Lemma for π_1, π_2, and Ψ provides $(b,b)\ \Psi\ (b,c)$, hence $(b,c) \in \mathrm{diag}(\underline{A})$ i.e. $b = c$.

As a particular example how to use the above theorem let us ask, in which algebras in an FP-variety, the set of solutions of a family of equations can be described by congruence classes (as in the familiar example of vector spaces). Simply look at the trivial equation $x = y$ whose set of solutions just consists of $\mathrm{diag}(\underline{A})$. Thus only modules over a commutative ring R remain, as in the modular case [17].

Similarly many results from CSAKANY [7] or [8], can directly be restated for FP-varieties.

Let us define an algebra to be hamiltonian, if every subalgebra is a class of some congruence relation.

Since $\mathrm{diag}(\underline{A})$ is always a subalgebra of $\underline{A} \times \underline{A}$ we get

11.9 Corollary: An algebra A in an FP-variety is affine, if and only if $\underline{A} \times \underline{A}$ is hamiltonian.

If we look at varieties of affine algebras, then we may even step outside the framework of FP-varieties. For this sake let us define:

An algebra $\underline{A}$ is called Jonsson-Tarski algebra, if it has a binary term $+$ and a constant term 0 such that the equations $x + 0 = 0 + x = x$ hold.

Thus Jonsson-Tarski algebras are just groupoids with unit with possibly some more operations added. A deep theory of decompositions was developed by JONSSON and TARSKI for those algebras in [30] (they required $\{0\}$ to be a subalgebra).

Clearly the examples of FP-varieties, cited at the beginning of this chapter are Jonsson-Tarski-varieties on defining

$$x + y := p(x,o,y).$$

We need a theorem due to KLUKOVITS [31]:

11.10 Theorem: A variety $\underline{V}$ is hamiltonian, if and only if for every term $f(x_1,\dots,x_n)$ there exists a ternary term h_f, such that the equation

$$f(x_1,\dots,x_n) = h_f(x_o, f(x_o,x_2,\dots,x_n), x_1)$$

holds.

Proof: Given f, look at the free $\underline{V}$-algebras over the generating set $\{x_0,x_1,\ldots,x_n\}$. Since the subalgebra $\underline{U}$, generated by $\{x_0,x_1,f(x_0,x_2,\ldots,x_n)\}$ must be a congruence class, and $f(x_0,x_2,\ldots,x_n)$, considered as unary algebraic function $\tau(x_0)$, throws an element of $\underline{U}$, namely x_0 back into $\underline{U}$, we need $\tau(x_1)$ to be inside $\underline{U}$. Thus $\tau(x_1)$ has to be the result of a term h_f applied to the generators of $\underline{U}$.

For the other direction suppose $\underline{S}$ to be a subalgebra of $\underline{A}$ and $\tau(x)$ a unary algebraic function on $\underline{A}$, i.e. $\tau(x) = f(x,a_2,\ldots,a_n)$. If $\tau(u) \in \underline{S}$ for some $u \in \underline{S}$, then for any other $u' \in \underline{S}$ we get $\tau(u') = h_f(u,\tau(u),u')$, an element of $\underline{S}$, thus $\underline{S}$ is a congruence class.

11.11 Theorem: Let $\underline{V}$ be a hamiltonian variety of Jonsson-Tarski algebras. Then $\underline{V}$ is polynomially equivalent to a variety of modules.

Proof: Set $p(x,y,z) := h_+(y,x,z)$.
Then

$$\begin{aligned} \underline{p(x,x,z)} &= h_+(x,x,z) = h_+(x,x+0,z) = \\ &= z+0 = \\ &= \underline{z} \end{aligned}$$

and

$$\begin{aligned} \underline{p(x,0,0)} &= h_+(0,x,0) = h_+(0,0+x,0) = \\ &= 0+x = \\ &= \underline{x}\ , \end{aligned}$$

bringing us back inside an FP-variety.

12. KRONECKER PRODUCTS

We have frequently been encountered with situations where some (or all) operations of an algebra $\underline{A}$ have properties which are commonly characteristic for homomorphisms. A typical example is theorem 9.1. The equation just says that every n-ary operation $f(x_1,\ldots,x_n)$ is an "n-ary homomorphism" with respect to $t(x,y,z)$. This statement is symmetric in the sense, that it can be read as $t(x,y,z)$ being a ternary homomorphism with regard to $f(x_1,\ldots,x_n)$. The importance of this property is evident in the proof of 9.2.

Secondly, the kernel of a homomorphism is a congruence relation. In particular, suppose $\phi: \underline{A}^n \longrightarrow \underline{A}$ is an n-ary homomorphism and consider the kernel of ϕ. Suppose $\phi(x,z_2,\ldots,z_n) = \phi(x,y_2,\ldots,y_n)$ for some $x,z_2,\ldots,z_n,y_2,\ldots,y_n$. Then the Shifting Lemma along factor congruence implies: $\phi(a,z_2,\ldots,z_n) = \phi(a,y_2,\ldots,y_n)$ for <u>every</u> $a \in \underline{A}$. Thus in an FP-variety this property is formally the same as the term condition from 6.7.

To put the discussion into a more general surrounding, let us look at general categories C having finite products and let us define an <u>algebra</u> (of type Δ) <u>in the category</u> C. This is a C-object $\underline{A}$ together with C-morphisms $f_i \in \mathrm{Hom}_C(\underline{A}^{n_i},\underline{A})$.

A $\underline{V}$-algebra <u>in</u> C must also satisfy the equations given gy the variety $\underline{V}$. Equations have to be expressed by commutative diagram unless the category is concrete.

For example, if T is the category of topological spaces, a $\underline{V}$-algebra <u>in</u> $\underline{T}$ is a topological $\underline{V}$-algebra where every operation is continuous. If $\underline{V}$ is idempotent then homotopy groups are <u>Groups in</u> $\underline{V}$, see TAYLOR [37].

If P is the category of posets then algebras <u>in</u> P are supposed to have all their operations order preserving. We will however only be concerned with the case where C is a variety $\underline{W}$ of algebras.

Thus let $\underline{A}$ be a $\underline{V}$-algebra <u>in</u> $\underline{W}$, f an n-ary $\underline{V}$-operation and g an m-ary $\underline{W}$-operation. Both f and g are defined on $\underline{A}$ but moreover f is a homomorphism (with respect to the $\underline{W}$-structure) from $\underline{A}^n$ to $\underline{A}$. Thus for elements $x_{ij} \in A$ with $1 \le i \le m$, $1 \le j \le n$ we have

$$f(g(x_{11},\ldots,x_{m1}),\ldots,g(x_{1n},\ldots,x_{mn})) = \\ = g(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{m1},\ldots,x_{mn})).$$

Obviously the $\underline{V}$-algebras <u>in</u> $\underline{W}$ form a variety of type $\Delta_V \cup \Delta_W$ which we denote by $\underline{V} \otimes \underline{W}$ and which we call the Kronecker product of $\underline{V}$ and $\underline{W}$.

Clearly $\underline{V} \otimes \underline{W} = \underline{W} \otimes \underline{V}$ and $\underline{V} \otimes (\underline{W} \otimes \underline{U}) = (\underline{V} \otimes \underline{W}) \otimes \underline{U}$. As an example how to deal with Kronecker products let us show

12.1 Proposition: (a) The Kronecker product of two permutable varieties is affine and (b) the Kronecker product of a permutable variety with a distributive variety is trivial.

Proof: (a) Let p, resp. q be the Mal'cev terms. Then

$$\begin{aligned} p(x,y,z) &= p(q(x,y,y),q(z,z,y),q(z,z,z)) = \\ &= q(p(x,z,z),p(y,z,z),q(y,y,z)) = \\ &= q(x,y,z). \end{aligned}$$

Hence $p=q$ is a Mal'cev term commuting with itself. Thus theorem 4.7 in [16] may be applied. Another way is by 9.1. For (b) let p be the Mal'cev term for $\underline{V}$ and q_i be the Jonsson terms for $\underline{W}$, $1 \le i \le n$. Let n be as small as possible such that Jonsson terms $q_o,\ldots,q_n$ exist. Depending on whether n is odd or even we calculate:

$$\begin{aligned} q_{n-1}(x,y,z) &= q_{n-1}(p(x,x,x),p(y,x,x),p(x,x,z)) = \\ &= p(q_{n-1}(x,y,x),q_{n-1}(x,x,x),q_{n-1}(x,x,z)) = \\ &= q_{n-1}(x,x,z) = z \end{aligned}$$

or

$$\begin{aligned} q_{n-1}(x,y,z) &= q_{n-1}(p(x,x,x),p(y,z,z),p(x,x,z)) = \\ &= p(q_{n-1}(x,y,x),q_{n-1}(x,z,x),q_{n-1}(x,z,z)) = \\ &= q_{n-1}(x,z,z) = z, \end{aligned}$$

in both cases contradicting the minimality of n.

If the terms defining the varieties $\underline{V}$ and $\underline{W}$ become more complicated, this method of determining $\underline{V} \otimes \underline{W}$ is too circumstantial. For example, how to set up the "right" equations for the Kronecker product of two modular varieties? How about two FP-varieties?

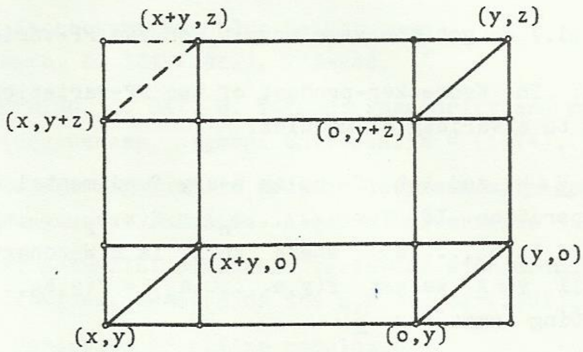
Here the geometric methods help us to circumvent the problem of setting up the appropriate equations.

The first example considers Jonsson-Tarski-algebras in FP-varieties and shows that they are abelian groups.
For modular varieties instead of FP-varieties the proof could have been given right after chapter 4. It illuminates the usefulness of "thinking in pictures". We make this clear by pointing out in every step, which geometric picture is responsible for which algebraic property.

12.2 Proposition: Let $+: \underline{A}^2 \longrightarrow \underline{A}$ be a homomorphism such that for some $o \in \underline{A}$, $o + x = x + o = x$. If $\underline{A}$ generates an FP-variety then $+$ is an abelian group operation.

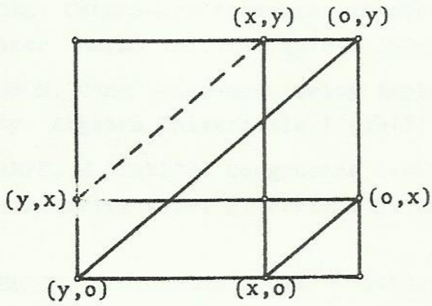
Proof: (i) + is cancellative (Shifting Lemma). If $a+x = a+y$, then (a,x) ker+ (a,y), hence (o,x) ker+ (o,y) by the Shifting Lemma, i.e. $x = o+x = o+y = y$. Similarly $x+a = y+a$ implies $x = y$. Note that (i) is equivalent to $\ker + \wedge \pi_1 = \ker + \wedge \pi_2 = 0$. We need this for

(ii) + is associative (Reidemeister Configuration, Cube Lemma). Since (y,z) ker+ $(o,y+z)$, (y,o) ker+ (o,y) and (x,y) ker+ $(o,x+y)$ we get from 11.5 that $(x+y,z)$ ker+ $(x,y+z)$ i.e. $(x+y)+z = x+(y+z)$.

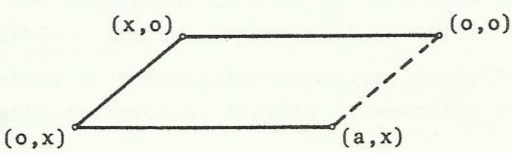


(iii) + is commutative (Escher Cube, resp. 11.6).

Since (o,y) ker+ (y,o), (o,x) ker+ (x,o) we get from 11.6 that (x,y) ker+ (y,x) i.e. $x+y = y+x$.



(iv) Existence of an inverse (Factor permutability).
Given (x,o) ker+ (o,x), factor permutability provides an element a with (o,o) ker+ (a,x) i.e. $a+x = o$.



As a corollary:

12.3 Corollary: The Kronecker product of a Jonsson-Tarski-variety with an FP-variety is polynomially equivalent to a variety of modules.

Proof: Suppose $\underline{V}$ is the Jonsson-Tarski-variety and $\underline{W}$ is the FP-variety. We know that $+$ is an abelian group operation commuting with every $\underline{W}$-operation. From the geometrical property of the W-terms $d(x,y,z)$ we see that in fact $d(x,y,z) = x - y + z$. Hence every other $\underline{V}$-operation commutes with $x - y + z$ too.

With the help of 11.7 we get the same result for two FP-varieties:

12.4 Proposition: The Kronecker-product of two FP-varieties is polynomially equivalent to a variety of modules.

Proof: Take $\underline{A} \in \underline{V} \otimes \underline{W}$ and let f be an n-ary fundamental operation. Say, f is a $\underline{V}$-operation. If $f(x,a_2,\ldots,a_n) = f(x,b_2,\ldots,b_n)$ then $(x,a_2,\ldots,a_n)$ kerf $(x,b_2,\ldots,b_n)$ where kerf is a $\underline{W}$-congruence relation. Hence for all $y \in \underline{A}$ we get $f(y,a_2,\ldots,a_n) = f(y,b_2,\ldots,b_n)$ because of the Shifting Lemma for $\underline{W}$.

Therefore diag($\underline{A}$) is a congruence class for the $\underline{V}$-operations and similarly for the $\underline{W}$-operations. With 11.7 and 11.8 the result follows.

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